ON THE NEW FORM OF BETHE ANSATZ EQUATIONS AND SEPARATION OF VARIABLES IN THE $\mathfrak{sl}_3$ GAUDIN MODEL

E. MUKHIN [*,1], V. SCHECHTMAN [**], V. TARASOV [*,⋆,2], AND A. VARCHENKO [⋆⋆,3]

[*] *Department of Mathematical Sciences, Indiana University – Purdue University, Indianapolis, 402 North Blackford St, Indianapolis, IN 46202-3216, USA*

[⋆] *St. Petersburg Branch of Steklov Mathematical Institute, Fontanka 27, St. Petersburg, 191023, Russia*

[**] *Laboratoire Emile Picard, UFR MIG, Université Paul Sabatier, 31062 Toulouse, France*

[⋆⋆] *Department of Mathematics, University of North Carolina at Chapel Hill, Chapel Hill, NC 27599-3250, USA*

Dedicated to V.I. Arnold on the occasion of his 70^{th} birthday.

Abstract. A new form of Bethe ansatz equations is introduced. A version of a separation of variables for the quantum $\mathfrak{sl}_3$ Gaudin model is presented.

1. Introduction

The separation of variables for the quantum $\mathfrak{sl}_2$ Gaudin model was constructed by Sklyanin in [Sk]. In this paper, we give an analogue of Sklyanin's construction for the Lie algebra $\mathfrak{sl}_3$.

We were inspired by Stoyanovsky's paper [St], in which the author uses Sklyanin's change of variables to establish a relation between the $\mathfrak{sl}_2$ Knizhnik-Zamolodchikov equations [KZ] and the Belavin-Polyakov-Zamolodchikov equations [BPZ], and to construct integral formulae for solutions to the BPZ equations.

1.1. The paper is organized as follows. In Section 2, we introduce notations and recall the definition of the KZ equations and Gaudin model.

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1.2. In Section 3, we recall the definition of the master function and canonical weight function. In Theorem 3.4.1, we recall the main fact of the Bethe ansatz method: if $\boldsymbol{t}$ is a critical point of the master function, then the value at $\boldsymbol{t}$ of the canonical weight function is an eigenvector of the Gaudin Hamiltonians, see [RV], cf. [Ba1, Ba2].

The critical point equations for the master function are called also the Bethe ansatz equations for the Gaudin model.

1.3. In Section 4, we discuss different forms of the Bethe ansatz equations for the Lie algebras $\mathfrak{sl}_2$ and $\mathfrak{sl}_3$.

For $\mathfrak{sl}_2$, there are two ways to describe solutions to the Bethe ansatz equations. The original way: a solution is a collection of numbers $\boldsymbol{t} = (t_1^{(1)}, \dots, t_{l_1}^{(1)})$, satisfying the critical point equations (12). The second way: given the numbers $z_1, \dots, z_n, (\Lambda_i, \alpha_1), (\Lambda_i, \Lambda_j)$, a solution is a polynomial $P(x) = \prod_{i=1}^{l_1} (t_i^{(1)} - x)$ of one variable and numbers $\mu_1, \dots, \mu_n$, $\mu_1 + \dots + \mu_n = 0$, satisfying the differential equation

$$P'' \; - \; P' \sum_{i=1}^{n} \frac{(\Lambda_i, \alpha_1)}{x - z_i} \; + \; P \sum_{i=1}^{n} \frac{1}{x - z_i} \, (\, \mu_i - \sum_{j \neq i} \frac{(\Lambda_i, \Lambda_j)}{z_i - z_j} \,) \; = 0 \; . \tag{1}$$

The equivalence of two descriptions is a classical fact which goes back to Stieltjes, see [Sti] and Sec. 6.8 in [Sz].

For $\mathfrak{sl}_3$, there are four ways to describe solutions to the Bethe ansatz equations. In this introduction, we mention only two of the four: the original way and the new way suggested in this paper.

The original way: a solution is a collection of numbers $\boldsymbol{t} = (t_1^{(1)}, \dots, t_{l_1}^{(1)}, t_1^{(2)}, \dots, t_{l_2}^{(2)})$, satisfying the critical point equations (15). The new way: given the numbers $z_1, \dots, z_n$, (Λ_i, α_1), (Λ_i, α_2), (Λ_i, Λ_j), a solution is a pair

$$P_1(x) \; = \; \prod_{i=1}^{l_1} (t_i^{(1)} - x) \; , \qquad P_2(x) \; = \; \prod_{i=1}^{l_2} (t_i^{(2)} - x)$$

of polynomials of one variable and a set of numbers $\mu_1, \dots, \mu_n$, $\mu_1 + \dots + \mu_n = 0$, satisfying the differential equation

$$\begin{aligned} P_1'' P_2 - P_1' P_2' + P_1 P_2'' - P_1' P_2 \sum_{i=1}^{n} \frac{(\Lambda_i, \alpha_1)}{x - z_i} - P_1 P_2' \sum_{i=1}^{n} \frac{(\Lambda_i, \alpha_2)}{x - z_i} + \\ + P_1 P_2 \sum_{i=1}^{n} \frac{1}{x - z_i} (\, \mu_i - \sum_{j \neq i} \frac{(\Lambda_i, \Lambda_j)}{z_i - z_j} \,) \; = 0 \; , \end{aligned} \tag{2}$$

see Theorem 4.2.3. This theorem is the first main result of our paper.

1.4. In Section 5, we describe Sklyanin's separation of variables for the $\mathfrak{sl}_2$ Gaudin model, following the exposition in [St].

The general problem in the Gaudin model is to find eigenfunctions of the commuting Gaudin Hamiltonians $H_i(\boldsymbol{z})$, $i = 1, \dots, n$, where $H_1(\boldsymbol{z}) + \dots + H_n(\boldsymbol{z}) = 0$. In suitable coordinates, the Hamiltonians are differential operators acting on polynomials in n variables $x^{(1)}, \dots, x^{(n)}$.

The eigenfunction equations are

$$H_i(\boldsymbol{z})\, F(x^{(1)}, \dots, x^{(n)}) \;=\; \mu_i\, F(x^{(1)}, \dots, x^{(n)}) \;, \qquad i = 1, \dots, n \;,$$

where the eigenvalues satisfy the equation $\mu_1 + \dots + \mu_n = 0$. The famous Sklyanin's change of variables (18) from variables $x^{(1)}, \dots, x^{(n)}$ to new variables $u, y^{(1)}, \dots, y^{(n-1)}$ transforms the eigenfunction equations to the following equations for the unknown polynomial $F(\boldsymbol{x}(u, \boldsymbol{y}))$,

$$(3) \quad \Big(-\partial^2_{y^{(j)}} \;+\; \sum_{i=1}^{n} \frac{(\Lambda_i, \alpha_1)}{y^{(j)} - z_i}\, \partial_{y^{(j)}} \;+\; \sum_{i=1}^{n} \frac{1}{y^{(j)} - z_i}\, \big(-\mu_i + \sum_{k \neq i} \frac{(\Lambda_i, \Lambda_k)}{z_i - z_k} \big) \Big) F \;=\; 0 \;,$$

$j = 1, \dots, n-1$, see [Sk, St] and Theorem 5.3.1.

This Sklyanin's statement has three interesting features.

The first is that the variables have separated: the j-th equation depends on variable $y^{(j)}$ only and does not depend on u and other variables $y^{(i)}$.

The second feature is that the differential operator is the same in all equations. Namely, the differential operator for an index j' can be obtained from the differential operator for the index j by replacing $y^{(j)}$ with $y^{(j')}$.

The third feature is that the differential operator in Sklyanin's equations is the same as the differential operator in the Bethe ansatz equation (1).

Sklyanin and Stoyanovsky also consider the canonical weight function. In Sklyanin's variables, the canonical weight function takes the form

$$\Psi(\boldsymbol{t}, \boldsymbol{z}, u, \boldsymbol{y}) \;=\; u^{l_1}\, \frac{P(y^{(1)}) \dots P(y^{(n-1)})}{P(z_1) \dots P(z_n)} \;,$$

where l_1 is a nonnegative integer and $P(x) = \prod_{i=1}^{l_1} (t_i^{(1)} - x)$.

The canonical weight function, as a function of u, $y^{(1)}$, $\dots$, $y^{(n-1)}$, is the product of functions of one variable. This is another manifestation of separation of variables.

Now if one looks for a value of the parameters $\boldsymbol{t}$ such that $\Psi(\boldsymbol{t}, \boldsymbol{z}, u, \boldsymbol{y})$, as a function of $u, \boldsymbol{y}$, becomes an eigenfunction of the Gaudin operators, then one gets the Bethe ansatz equation (1) for the unknown polynomial $P(x)$.

Hence, if for some $\boldsymbol{t}$, the function $\Psi(\boldsymbol{t}, \boldsymbol{z}, u, \boldsymbol{y})$ is an eigenfunction, then $\boldsymbol{t}$ satisfies the Bethe ansatz equations.

1.5. In Section 6, we describe an analog of Sklyanin's statements for $\mathfrak{sl}_3$.

Again the general problem is to find eigenfunctions of the Gaudin Hamiltonians $H_i(\boldsymbol{z})$, $i = 1, \dots, n$, where $H_1(\boldsymbol{z}) + \dots + H_n(\boldsymbol{z}) = 0$. Now the Hamiltonians are differential operators acting on polynomials in $3n$ variables $x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, \dots, x_1^{(n)}, x_2^{(n)}, x_3^{(n)}$.

The eigenfunction equations are

$$H_i(\boldsymbol{z})\, F(\boldsymbol{x}) \;=\; \mu_i\, F(\boldsymbol{x})\;, \qquad i=1,\dots,n\;,$$

where the eigenvalues satisfy the equation $\mu_1+\dots+\mu_n=0$.

We make a change from variables $x_1^{(1)},x_2^{(1)},x_3^{(1)},\;\dots,\;x_1^{(n)},x_2^{(n)},x_3^{(n)}$ to new variables $u_1,u_2,u_3,y_1^{(1)},y_2^{(1)},y_3^{(1)},\dots,y_1^{(n-1)},y_2^{(n-1)},y_3^{(n-1)}$, see formula (27) analogous to Sklyanin's formula.

We define the degree of a polynomial in $\mathbb{C}[\boldsymbol{u},\boldsymbol{y}]$ as its degree with respect to variable u_3.

On the affine space with coordinates $\boldsymbol{u},\boldsymbol{y}$, we consider the affine subspace

$$\mathfrak{D} \;=\; \{(\boldsymbol{u},\boldsymbol{y}) \mid y_1^{(j)}=y_2^{(j)},\; j=1,\dots,n-1\}\;.$$

It turns out that the Gaudin Hamiltonians have the decomposition

$$H_i(\boldsymbol{z}) = \;\bar H_i(\boldsymbol{z})_0 + H_i(\boldsymbol{z})_{>0} + H_i(\boldsymbol{z})_{<0}\;,$$

where the operator $\bar H_i(\boldsymbol{z})_0$ preserves the degree, the operator $H_i(\boldsymbol{z})_{>0}$ increases the degree by one, and the operator $H_i(\boldsymbol{z})_{<0}$ decreases the degree by one. Moreover, it turns out that for any $F\in\mathbb{C}[\boldsymbol{u},\boldsymbol{y}]$, the restriction to $\mathfrak{D}$ of the polynomial $H_i(\boldsymbol{z})_{<0}F$ is zero.

These remarks show that the eigenfunction equations become "upper triangular" with respect to the degree decomposition after restriction to $\mathfrak{D}$.

Let $F(\boldsymbol{u},\boldsymbol{y})$ be an unknown eigenfunction with eigenvalues $\mu_1,\dots,\mu_n$. Let $F(\boldsymbol{u},\boldsymbol{y}) = F(\boldsymbol{u},\boldsymbol{y})_0 + F(\boldsymbol{u},\boldsymbol{y})_1+\dots$ be the degree decomposition. According to our change of variables,

$$F(\boldsymbol{u},\boldsymbol{y})_0 \;=\; u_1^{l_1}u_2^{l_2}\, f(y_1^{(1)},y_2^{(1)},\dots,y_1^{(n-1)},y_2^{(n-1)})\;,$$

where $l_1,\ l_2$ are nonnegative integers and f is a polynomial depending on variables $y_1^{(j)},y_2^{(j)}$, $j=1,\dots,n-1$, only.

The triangularity remark implies the following equations for the degree zero part of the eigenfunction,

$$\Big((\bar H_i(\boldsymbol{z})_0-\mu_i)\,F_0\Big)\Big|_{\mathfrak{D}} \;=\; 0\;, \qquad i=1,\dots,n\;. \tag{4}$$

1.5.1. **Theorem.** *After suitable renormalization, equations (4) take the form*

$$\begin{aligned}\Big(-\frac{\partial^2 f}{\partial y_1^{(j)}\partial y_1^{(j)}} \;+\; \frac{\partial^2 f}{\partial y_1^{(j)}\partial y_2^{(j)}} \;-\; \frac{\partial^2 f}{\partial y_2^{(j)}\partial y_2^{(j)}} \;-\; \frac{\partial f}{\partial y_1^{(j)}}\sum_{i=1}^n \frac{(\Lambda_i,\alpha_1)}{y_1^{(j)}-z_i} \;-&\\ -\;\frac{\partial f}{\partial y_2^{(j)}}\sum_{i=1}^n \frac{(\Lambda_i,\alpha_2)}{y_1^{(j)}-z_i} \;+\; f\sum_{i=1}^n \frac{1}{y_1^{(j)}-z_i}\,\big(-\mu_i+\sum_{k\neq i}\frac{(\Lambda_i,\Lambda_k)}{z_i-z_k}\,\big)\Big)\Big|_{\mathfrak{D}} \;=\; 0\;.&\end{aligned} \tag{5}$$

This is our second main result, see Theorem 6.5.4.

As Sklyanin's equations (3) for $\mathfrak{sl}_2$, our equations (5) have three interesting properties.

The first is that the variables have separated: the j-th equation depends on variables $y_1^{(j)}$, $y_2^{(j)}$ only and does not depend on $\boldsymbol{u}$ and other variables $y_1^{(j')}$, $y_2^{(j')}$.

The second property is that the differential operator is the same in all equations. Namely, the differential operator for an index j' can be obtained from the differential operator for the index j by replacing $y_1^{(j)}, y_2^{(j)}$ with $y_1^{(j')}, y_2^{(j')}$, respectively.

The third property is that the differential operator in equations (5) is the same as the differential operator in the Bethe ansatz equation (2).

Then we consider the canonical weight function $\Psi(\boldsymbol{t}, \boldsymbol{z}, \boldsymbol{u}, \boldsymbol{y})$ and its weight decomposition $\Psi = \Psi_0 + \Psi_1 + \dots$. It turns our that the degree zero term has the form

$$\Psi(\boldsymbol{t}, \boldsymbol{z}, \boldsymbol{u}, \boldsymbol{y})_0 \; = \; u_1^{l_1} u_2^{l_1} \, \frac{\prod_{j=1}^{n-1} P_1(y_1^{(j)}) P_2(y_2^{(j)})}{\prod_{s=1}^{n} P_1(z_s) P_2(z_s)} \; , \tag{6}$$

where l_1, l_2 are nonnegative integers, $P_1(x) = \prod_{i=1}^{l_1}(t_i^{(1)} - x)$, $P_2(x) = \prod_{i=1}^{l_2}(t_i^{(2)} - x)$. Moreover, it turns out that the higher degree terms of Ψ are uniquely determined by the degree zero term Ψ_0, see Theorem 6.6.3 and Section 6.7.3.

According to formula (6), the degree zero term Ψ_0 of the canonical weight function is the product of functions of one variable. This is another manifestation of separation of variables.

Now assume that one looks for a value of parameters $\boldsymbol{t}$ such that $\Psi(\boldsymbol{t}, \boldsymbol{z}, \boldsymbol{u}, \boldsymbol{y})$, as a function of $\boldsymbol{u}, \boldsymbol{y}$, becomes an eigenfunction of the Gaudin operators. Then, by Theorem 1.5.1, one gets the Bethe ansatz equation (2) for the unknown polynomials $P_1(x)$, $P_2(x)$.

Hence, if for some $\boldsymbol{t}$, the function $\Psi(\boldsymbol{t}, \boldsymbol{z}, \boldsymbol{u}, \boldsymbol{y})$ is an eigenfunction, then $\boldsymbol{t}$ satisfies the Bethe ansatz equations. The precise statement see in Theorem 6.7.4.

1.6. In the appendix we consider the Bethe ansatz equations for the Gaudin model associated with an arbitrary Kac-Moody algebra of rank r. We introduce a polylinear differential equation for a collection $P_1(x), \dots, P_r(x)$ of polynomials of one variable and a collection of numbers $\mu_1, \dots, \mu_n$, $\mu_1 + \dots + \mu_n = 0$. That equation is an analog of the differential equation (2). Under certain conditions, we show that the union of roots of polynomials $P_1(x), \dots, P_r(x)$ form a solution to the Bethe ansatz equations if and only if $P_1(x), \dots, P_r(x)$ and $\mu_1, \dots, \mu_n$ satisfy that polylinear differential equation.

This fact may be considered as a generalization of Stieltjes' Lemma 4.1.2 (see also [Sti], and Sec. 6.8 in [Sz]) to an arbitrary Kac-Moody algebra.

1.7. In the next paper we plan to extend the results of this paper from $\mathfrak{sl}_3$ to other Lie algebras.

1.8. This work has been started when the fourth author visited Université Paul Sabatier in Toulouse in May-June of 2006. He thanks the university for hospitality.

2. KZ equations and the Gaudin model

2.1. **Lie algebra $\mathfrak{sl}_{r+1}$.** Consider the Lie algebra $\mathfrak{gl}_{r+1}$ with standard generators $e_{a,b}$, $a,b=1,\dots,r+1$. Set $h_a = e_{a,a} - e_{a+1,a+1}$ for $a=1,\dots,r$. Consider the Lie subalgebra

$$\mathfrak{g} = \mathfrak{sl}_{r+1}$$

generated by elements $e_{a,b}$, $a \neq b$, and h_a, $a=1,\dots,r$. Let $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$,

$$\mathfrak{n}_- = \oplus_{a>b}\, \mathbb{C}\cdot e_{a,b}\ , \qquad \mathfrak{h} = \oplus_{a=1}^{r}\, \mathbb{C}\cdot h_a\ , \qquad \mathfrak{n}_+ = \oplus_{a<b}\, \mathbb{C}\cdot e_{a,b}\ ,$$

be the Cartan decomposition. Let $\alpha_a \in \mathfrak{h}^*$, $a=1,\dots,r$, be simple roots.

Fix the invariant scalar product on $\mathfrak{g}$ such that $(h_i,h_i) = 2$. The scalar product identifies $\mathfrak{g}$ and its dual $\mathfrak{g}^*$.

2.1.1. The Casimir element $\Omega \in \mathfrak{g}^{\otimes 2}$ is the element $\sum_i x_i \otimes x_i$, where (x_i) is an orthonormal basis of $\mathfrak{g}$.

2.1.2. For $\mathfrak{g} = \mathfrak{sl}_2$,

$$\Omega = e_{2,1}\otimes e_{1,2} + e_{1,2}\otimes e_{2,1} + \frac{1}{2} h_1 \otimes h_1\ .$$

For $\mathfrak{g} = \mathfrak{sl}_3$,

$$\begin{aligned}\Omega = {} & e_{2,1}\otimes e_{1,2} + e_{1,2}\otimes e_{2,1} + \\ & e_{3,2}\otimes e_{2,3} + e_{2,3}\otimes e_{3,2} + e_{3,1}\otimes e_{1,3} + e_{1,3}\otimes e_{3,1} + \\ & h_1 \otimes \Big(\frac{2}{3}h_1 + \frac{1}{3}h_2\Big) + h_2 \otimes \Big(\frac{1}{3}h_1 + \frac{2}{3}h_2\Big)\ .\end{aligned}$$

2.1.3. Let $U(\mathfrak{n}_-)$ be the universal enveloping algebra of $\mathfrak{n}_-$,

$$U(\mathfrak{n}_-) = \oplus_{\boldsymbol{l}\in\mathbb{Z}^r_{\geq 0}}\, U(\mathfrak{n}_-)[\boldsymbol{l}]\ ,$$

where for $\boldsymbol{l} = (l_1,\dots,l_r)$, the space $U(\mathfrak{n}_-)[\boldsymbol{l}]$ consists of elements f such that

$$[f,h] = \langle h\,, \sum_{i=1}^{r} l_i\alpha_i\rangle f\ .$$

The element $\prod_i e_{a_i,b_i}$ with $a_i > b_i$ belongs to the graded subspace $U(\mathfrak{n}_-)\,[\boldsymbol{l}]$, where $\boldsymbol{l} = \sum_i \boldsymbol{l}^{(i)}$ with

$$\boldsymbol{l}^{(i)} = (0,0,\dots,0,1_b,1_{b+1},\dots,1_{a-1},0,0,\dots,0)\ .$$

Fix an order on the set of elements $e_{a,b}$ with $r+1 \geq a > b \geq 1$. Set $e_{a,b} < e_{a',b'}$ if $b < b'$ or $b = b'$ and $a < a'$.

For example $e_{2,1} < e_{3,1} < e_{3,2}$.

Then the ordered products $\prod_{a>b} e_{a,b}^{n_{a,b}}$ form a graded PBW basis of $U(\mathfrak{n}_-)$.

2.1.4. The grading of $U(\mathfrak{n}_-)$ induces the grading of $U(\mathfrak{n}_-)^{\otimes n}$ for any n,

$$U(\mathfrak{n}_-)^{\otimes n} \;=\; \oplus_{\boldsymbol{l}}\, U(\mathfrak{n}_-)^{\otimes n}[\boldsymbol{l}]\ ,$$

where $U(\mathfrak{n}_-)^{\otimes n}[\boldsymbol{l}] = \oplus_{\boldsymbol{l}^{(1)}+\dots+\boldsymbol{l}^{(n)}=\boldsymbol{l}} U(\mathfrak{n}_-)[\boldsymbol{l}^{(1)}]\otimes\dots\otimes U(\mathfrak{n}_-)[\boldsymbol{l}^{(n)}]$. The PBW basis of tensor factors induces a graded PBW basis of $U(\mathfrak{n}_-)^{\otimes n}$.

2.1.5. For $\Lambda \in \mathfrak{h}^*$, denote by M_Λ the $\mathfrak{sl}_{r+1}$ Verma module with highest weight Λ. Denote by $v_\Lambda \in M_\Lambda$ its highest weight vector.

For $\boldsymbol{\Lambda} = (\Lambda_1, \dots, \Lambda_n)$, $\Lambda_s \in \mathfrak{h}^*$, denote

$$M_{\boldsymbol{\Lambda}} \;=\; M_{\Lambda_1} \otimes \dots \otimes M_{\Lambda_n}\ .$$

We have the weight decomposition

$$M_{\boldsymbol{\Lambda}} \;=\; \oplus_{\boldsymbol{l}\in\mathbb{Z}^r_{\geq 0}}\, M_{\boldsymbol{\Lambda}}[\boldsymbol{l}]\ ,$$

where $M_{\boldsymbol{\Lambda}}[(l_1, \dots, l_r)]$ is the subspace of vectors of weight $\sum_{s=1}^n \Lambda_s - \sum_{i=1}^r l_i\alpha_i$.

The PBW basis of $U(\mathfrak{n}_-)^{\otimes n}$ induces a graded PBW basis of $M_{\boldsymbol{\Lambda}}$.

Let

$$\mathrm{Sing}\, M_{\boldsymbol{\Lambda}}[\boldsymbol{l}] \;\subset\; M_{\boldsymbol{\Lambda}}[\boldsymbol{l}]$$

denote the subspace of singular vectors, i.e. the subspace of vectors annihilated by $\mathfrak{n}_+$.

2.2. **KZ equations.** The KZ equations on an $M_{\boldsymbol{\Lambda}}$-valued function $I(z_1, \dots, z_n)$ of complex variables $\boldsymbol{z} = (z_1, \dots, z_n)$ is the following system of differential equations

$$\kappa\, \frac{\partial I}{\partial z_i} \;=\; \sum_{j\neq i} \frac{\Omega^{(ij)}}{z_i - z_j}\, I\ , \qquad i = 1, \dots, n\ ,$$

where $\kappa \in \mathbb{C}^*$ is a parameter of the equations.

The KZ equations will not be used in this paper, but it is useful to keep them in mind when discussing the Gaudin model.

2.3. **Gaudin model.** Fix pairwise distinct complex numbers $z_1, \dots, z_n$. Denote

$$H_i(\boldsymbol{z}) \;=\; \sum_{j\neq i} \frac{\Omega^{(ij)}}{z_i - z_j}\ , \qquad i = 1, \dots, n\ .$$

These are linear operators on $M_{\boldsymbol{\Lambda}}$ called *the Gaudin Hamiltonians.*

The Gaudin Hamiltonians commute, $[H_i(\boldsymbol{z}), H_j(\boldsymbol{z})] = 0$ for $i \neq j$.

The Gaudin Hamiltonians commute with the action of $\mathfrak{sl}_{r+1}$ on $M_{\boldsymbol{\Lambda}}$ and hence they preserve the subspaces $\mathrm{Sing}\, M_{\boldsymbol{\Lambda}}[\boldsymbol{l}]$.

We also have $H_1(\boldsymbol{z}) + \dots + H_n(\boldsymbol{z}) = 0$.

3. Master function, canonical weight function, and the Bethe ansatz

3.1. **Master function,** [SV]. Let $\boldsymbol{\Lambda} = (\Lambda_1, \dots, \Lambda_n)$, $\Lambda_s \in \mathfrak{h}^*$, be a collection of $\mathfrak{sl}_{r+1}$-weights and $\boldsymbol{l} = (l_1, \dots, l_r)$ a collection of nonnegative integers. Set $l = l_1 + \dots + l_r$. Introduce a function of n variables $\boldsymbol{z} = (z_1, \dots, z_n)$ and l variables

$$\boldsymbol{t} = (t_1^{(1)}, \dots, t_{l_1}^{(1)}, \dots, t_1^{(r)}, \dots, t_{l_r}^{(r)})$$

by the formula

$$\Phi(\boldsymbol{t};\boldsymbol{z};\boldsymbol{\Lambda};\boldsymbol{l}) \;=\; \prod_{1\le i<j\le n} (z_i - z_j)^{(\Lambda_i,\Lambda_j)} \;\times \tag{7}$$

$$\prod_{i=1}^{r}\prod_{j=1}^{l_i}\prod_{s=1}^{n}(t_j^{(i)} - z_s)^{-(\Lambda_s,\alpha_i)} \prod_{i=1}^{r}\prod_{1\le j<s\le l_i}(t_j^{(i)} - t_s^{(i)})^2 \prod_{i=1}^{r-1}\prod_{j=1}^{l_i}\prod_{k=1}^{l_{i+1}}(t_j^{(i)} - t_k^{(i+1)})^{-1} .$$

The function Φ is a (multi-valued) function of $\boldsymbol{t}$, depending on parameters $\boldsymbol{z}, \boldsymbol{\Lambda}$. The function Φ is called *the master function.*

3.1.1. The product of symmetric groups

$$S_{\boldsymbol{l}} \;=\; S_{l_1} \times \dots \times S_{l_r}$$

acts on variables $\boldsymbol{t}$ by permuting the variables with the same upper index. The master function is $S_{\boldsymbol{l}}$-invariant.

3.1.2. A point $\boldsymbol{t}$ with complex coordinates will be called *a critical point* of $\Phi(\,\cdot\,;\boldsymbol{z};\boldsymbol{\Lambda};\boldsymbol{l})$ if the following system of l equations is satisfied

$$\left(\Phi^{-1}\frac{\partial\Phi}{\partial t_j^{(i)}}\right)(\boldsymbol{t};\boldsymbol{z};\boldsymbol{\Lambda};\boldsymbol{l}) \;=\; 0\;, \qquad i = 1, \dots, r,\; j = 1, \dots l_i\;. \tag{8}$$

3.1.3. Equations (8) can be reformulated as the system of equations

$$\sum_{s=1}^{n}\frac{(\Lambda_s,\alpha_1)}{t_j^{(1)} - z_s} - \sum_{s=1,\; s\neq j}^{l_1}\frac{2}{t_j^{(1)} - t_s^{(1)}} + \sum_{s=1}^{l_2}\frac{1}{t_j^{(1)} - t_s^{(2)}} = 0\;, \tag{9}$$

$$\sum_{s=1}^{n}\frac{(\Lambda_s,\alpha_i)}{t_j^{(i)} - z_s} - \sum_{s=1,\; s\neq j}^{l_i}\frac{2}{t_j^{(i)} - t_s^{(i)}} + \sum_{s=1}^{l_{i-1}}\frac{1}{t_j^{(i)} - t_s^{(i-1)}} + \sum_{s=1}^{l_{i+1}}\frac{1}{t_j^{(i)} - t_s^{(i+1)}} = 0\;,$$

$$\sum_{s=1}^{n}\frac{(\Lambda_s,\alpha_r)}{t_j^{(r)} - z_s} - \sum_{s=1,\; s\neq j}^{l_r}\frac{2}{t_j^{(r)} - t_s^{(r)}} + \sum_{s=1}^{l_{r-1}}\frac{1}{t_j^{(r)} - t_s^{(r-1)}} = 0\;,$$

where $j = 1, \dots, l_1$ in the first group of equations, $i = 2, \dots, r-1$ and $j = 1, \dots, l_i$ in the second group of equations, $j = 1, \dots, l_r$ in the last group of equations.

3.2. **Canonical weight function,** [SV, RSV]**.** The PBW-basis of $U(\mathfrak{n}_-)$, defined in Section 2.1.3, is

$$\left\{ F_{I_0} \ = \ e_{2,1}^{i_{2,1}} \cdots e_{r+1,r}^{i_{r+1,r}} \right\} \ ,$$

where $I_0 = \{i_{a,b}\}_{a>b}$ runs over all sequences of nonnegative integers.

Let $F_{I_1}, \dots, F_{I_n}$ be elements of the standard PBW-basis, $I_j = \{i^j_{2,1}, \dots, i^j_{r+1,r}\}$. Set $I = (I_1, \dots, I_n)$. The corresponding basis vector

$$F_I v \ = \ F_{I_1} v_{\Lambda_1} \otimes \cdots \otimes F_{I_{n+1}} v_{\Lambda_n}$$

lies in $M_{\boldsymbol{\Lambda}}[\boldsymbol{l}]$ if

$$\sum_{j=1}^{n} \sum_{b=1}^{i} \sum_{a=i+1}^{r+1} i^j_{a,b} \ = \ l_i \ , \qquad \text{for all} \quad i = 1, \dots, r \ . \tag{10}$$

Denote by $P(\boldsymbol{l}, n)$ the set of all indices I corresponding to basis vectors in $M_{\boldsymbol{\Lambda}}[\boldsymbol{l}]$. The set $\{F_I v\}_{I \in P(\boldsymbol{l},n)}$ forms a basis of $M_{\boldsymbol{\Lambda}}[\boldsymbol{l}]$.

3.2.1. For $I \in P(\boldsymbol{l}, n)$, define the set

$$S(I) \ = \ \{ \ (j, a, b, q) \mid 1 \le j \le n, \quad 1 \le b < a \le r+1, \qquad 1 \le q \le i^j_{a,b} \ \} \ .$$

For $i = 1, \dots, r$, define the subset

$$S_i(I) \ = \ \{ \ s = (j, a, b, q) \in S(I) \mid b \le i < a \ \} \ .$$

Condition (10) implies $|S_i(I)| \ = \ l_i$ for $i = 1, \dots, r$.

Define the set

$$B(I) \ = \ \{ \ \beta = (\beta_1, \dots, \beta_r) \mid \text{for } i = 1, \dots, r, \ \beta_i \text{ is a bijection } S_i(I) \to \{1, \dots, l_i\} \ \} \ .$$

We have $|B(I)| = l_1! \cdots l_r!$.

For $s = (j, k, l, q) \in S(I)$ and $\beta \in B(I)$, introduce the rational function

$$\omega_{s,\beta} \quad = \quad \frac{1}{t^{(b)}_{\beta_b(s)} - z_j} \prod_{i=b+1}^{a-1} \frac{1}{t^{(i)}_{\beta_i(s)} - t^{(i-1)}_{\beta_{i-1}(s)}} \ .$$

Introduce the rational functions

$$\omega_I \ = \ \frac{1}{\prod_{a>b} \, (i^j_{a,b})!} \sum_{\beta \in B(I)} \prod_{s \in S(I)} \omega_{s,\beta} \ , \qquad \omega_{\boldsymbol{l},n} \ = \ \sum_{I \in P(\boldsymbol{l},n)} \omega_I \, F_I v \ .$$

The function $\omega_{\boldsymbol{l},n}$ defines a rational map

$$\omega_{\boldsymbol{l},n} \ : \ \mathbb{C}^l \times \mathbb{C}^n \ \to \ M_{\boldsymbol{\Lambda}}[\boldsymbol{l}] \ ,$$

called *the canonical weight function*.

3.2.2. The canonical weight function was introduced in [SV]. The formula for the canonical weight function, presented in Section 3.2.1, is proved in [RSV].

3.2.3. **Examples.** Let $n=2$. If $\boldsymbol{l}=(1,1,0,\dots,0)$, then

$$\begin{aligned}\omega_{\boldsymbol{l},n}(\boldsymbol{t},\boldsymbol{z}) &= \frac{1}{(t_1^{(1)}-z_1)(t_1^{(2)}-z_1)}\ e_{2,1}e_{3,2}v_{\Lambda_1}\otimes v_{\Lambda_2}+\frac{1}{(t_1^{(2)}-t_1^{(1)})(t_1^{(1)}-z_1)}\ e_{3,1}v_{\Lambda_1}\otimes v_{\Lambda_2}\\ &+ \frac{1}{(t_1^{(1)}-z_1)(t_1^{(2)}-z_2)}\ e_{2,1}v_{\Lambda_1}\otimes e_{3,2}v_{\Lambda_2}+\frac{1}{(t_1^{(2)}-z_1)(t_1^{(1)}-z_2)}\ e_{3,2}v_{\Lambda_1}\otimes e_{2,1}v_{\Lambda_2}\\ &+ \frac{1}{(t_1^{(1)}-z_2)(t_1^{(2)}-z_2)}\ v_{\Lambda_1}\otimes e_{2,1}e_{3,2}v_{\Lambda_2}+\frac{1}{(t_1^{(2)}-t_1^{(1)})(t_1^{(1)}-z_2)}\ v_{\Lambda_1}\otimes e_{3,1}v_{\Lambda_2}\ .\end{aligned}$$

If $\boldsymbol{l}=(2,0,\dots,0)$, then

$$\begin{aligned}\omega_{\boldsymbol{l},n}(\boldsymbol{t},\boldsymbol{z}) &= \frac{1}{(t_1^{(1)}-z_1)(t_2^{(1)}-z_1)}\ e_{2,1}^2v_{\Lambda_1}\otimes v_{\Lambda_2}\\ &+ \left(\frac{1}{(t_1^{(1)}-z_1)(t_2^{(1)}-z_2)}+\frac{1}{(t_2^{(1)}-z_1)(t_1^{(1)}-z_2)}\right)\ e_{2,1}v_{\Lambda_1}\otimes e_{2,1}v_{\Lambda_2}\\ &+ \frac{1}{(t_1^{(1)}-z_2)(t_2^{(1)}-z_2)}\quad v_{\Lambda_1}\otimes e_{2,1}^2v_{\Lambda_2}\ .\end{aligned}$$

3.3. **Hypergeometric solutions to the KZ equations.** The master function and canonical weight function were introduced in [SV] to solve the KZ equations. The hypergeometric solutions to the KZ equations with values in $\mathrm{Sing}\, M_{\boldsymbol{\Lambda}}[\boldsymbol{l}]$ have the form

$$I(\boldsymbol{z})\ =\ \int_{\gamma(\boldsymbol{z})}\Phi(\boldsymbol{t},\boldsymbol{z},\boldsymbol{\Lambda},\boldsymbol{l})^{1/\kappa}\ \omega_{\boldsymbol{l},n}(\boldsymbol{t},\boldsymbol{z})\ d\boldsymbol{t}\ ,$$

where $d\boldsymbol{t}\ =\ \wedge_{i,j}\,dt_j^{(i)}$ and $\gamma(\boldsymbol{z})\ \in\ \mathbb{C}^l\times\{\boldsymbol{z}\}$ is a horizontal family of l-dimensional cycles of the twisted homology defined by the multi-valued function $(\Phi)^{\frac{1}{\kappa}}$, see [SV, V].

In this paper, we will not use the hypergeometric solutions to the KZ equations, but the Bethe ansatz for the Gaudin model can be developed studying quasi-classical asymptotics of these solutions, [RV].

3.4. **Bethe ansatz for the Gaudin model.** For given $\boldsymbol{l}$ and distinct numbers $z_1,\dots,z_n$, the problem is to diagonalize simultaneously the Gaudin Hamiltonians, restricted to the subspace $\mathrm{Sing}\, M_{\boldsymbol{\Lambda}}[\boldsymbol{l}]$.

3.4.1. **Theorem** ([RV], cf. [Ba1, Ba2]). *Assume that $\boldsymbol{z}\in\mathbb{C}^n$ has distinct coordinates. Assume that $\boldsymbol{t}\in\mathbb{C}^l$ is a critical point of the master function $\Phi(\,\cdot\,,\boldsymbol{z},\boldsymbol{\Lambda},\boldsymbol{l})$. Then the vector $\omega_{\boldsymbol{l},n}(\boldsymbol{t},\boldsymbol{z})$ belongs to* $\mathrm{Sing}\, M_{\boldsymbol{\Lambda}}[\boldsymbol{l}]$ *and is an eigenvector of the Gaudin Hamiltonians with eigenvalues given by the derivatives of the logarithm of the master function:*

$$H_i(\boldsymbol{z})\ \omega_{\boldsymbol{l},n}(\boldsymbol{t},\boldsymbol{z})\ =\ (\frac{\partial}{\partial z_i}\log\,\Phi(\boldsymbol{t},\boldsymbol{z},\boldsymbol{\Lambda},\boldsymbol{l}))\ \omega_{\boldsymbol{l},n}(\boldsymbol{t},\boldsymbol{z})\ ,\qquad i=1,\dots,n\ . \tag{11}$$

This theorem was proved in [RV] using the quasi-classical asymptotics of the hypergeometric solutions of the KZ equations. The theorem also follows directly from Theorem 6.16.2 in [SV], cf. Theorem 7.2.5 in [SV].

3.4.2. Notice that a priori the vector $\omega_{\boldsymbol{l},n}(\boldsymbol{t},\boldsymbol{z})$ belongs to $M_{\boldsymbol{\Lambda}}[\boldsymbol{l}]$, but if $\boldsymbol{t}$ is a critical point, then $\omega_{\boldsymbol{l},n}(\boldsymbol{t},\boldsymbol{z})$ belongs to $\operatorname{Sing} M_{\boldsymbol{\Lambda}}[\boldsymbol{l}]$.

3.4.3. The critical point equations (9) are called *the Bethe ansatz equations.*

The values of the canonical weight function at the critical points (with respect to $\boldsymbol{t}$) of the master function are called *the Bethe vectors.*

4. Different forms of the Bethe ansatz equations for $\mathfrak{sl}_2$ and $\mathfrak{sl}_3$

4.1. **Bethe ansatz equations for $\mathfrak{sl}_2$.** The Bethe ansatz equations (9) for $\mathfrak{sl}_2$ take the form

$$\sum_{s=1}^{n} \frac{(\Lambda_s,\alpha_1)}{t_i^{(1)}-z_s} - \sum_{j=1,\ j\neq i}^{l_1} \frac{2}{t_i^{(1)}-t_j^{(1)}} = 0\,, \qquad i=1,\dots,l_1\,. \tag{12}$$

4.1.1. Introduce polynomials

$$F(x) = \prod_{s=1}^{n}(x-z_s)\,,\quad G(x) = F(x)\sum_{s=1}^{n}\frac{(\Lambda_s,\alpha_1)}{x-z_s}\,,\quad P(x)=\prod_{i=1}^{l_1}(t_i^{(1)}-x)\,.$$

We have $\deg F = n,\ \deg G = n-1,\ \deg\, P = l_1$.

4.1.2. **Lemma.** *Assume that the roots of P are simple. Assume that for any s we have $z_s \notin \{t_1^{(1)},\dots,t_{l_1}^{(1)}\}$.*

Then the roots of P form a solution to system (12) if and only if the polynomial $FP''-GP'$ is divisible by the polynomial P.

In other words, the roots of P form a solution to the Bethe ansatz equations (12) if and only if there exists a polynomial H of degree not greater than $n-2$ such that P is a solution to the differential equation

$$F\,P'' - G\,P' + H\,P = 0\,. \tag{13}$$

The lemma is a classical result due to Stieltjes, see [Sti] and Section 6.8 in [Sz].

4.1.3. **Lemma.** *There exist unique numbers $\mu_1,\dots,\mu_n$, $\mu_1+\dots+\mu_n=0$, such that*

$$\frac{H(x)}{F(x)} = \sum_{j=1}^{n}\frac{1}{x-z_j}\,(\,\mu_j - \sum_{k\neq j}\frac{(\Lambda_j,\Lambda_k)}{z_j-z_k}\,)\,. \qquad \square$$

4.1.4. **Corollary.** *Assume that the roots of P are simple. Assume that for any s we have $z_s \notin \{t_1^{(1)}, \dots, t_{l_1}^{(1)}\}$. Then the roots of P form a solution to the Bethe ansatz equations (12) if and only if there exist unique numbers $\mu_1, \dots, \mu_n$, $\mu_1 + \dots + \mu_n = 0$, such that P is a solution to the differential equation*

$$P'' \;-\; P' \sum_{i=1}^{n} \frac{(\Lambda_i, \alpha_1)}{x - z_i} \;+\; P \sum_{j=1}^{n} \frac{1}{x - z_j} \,(\, \mu_j - \sum_{k \neq j} \frac{(\Lambda_j, \Lambda_k)}{z_j - z_k} \,) \;=0\;. \tag{14}$$

4.1.5. The corollary provides two ways to describe solutions to the $\mathfrak{sl}_2$ Bethe ansatz equations.

The original way: a solution is a collection $\boldsymbol{t} = (t_1^{(1)}, \dots, t_{l_1}^{(1)})$ satisfying (12). The second way: a solution is a polynomial P and numbers $\mu_1, \dots, \mu_n$, $\mu_1 + \dots + \mu_n = 0$, satisfying (14).

4.1.6. **Lemma.** *For given F, G, H, if at least one of the numbers $(\Lambda_1, \alpha_1), \dots, (\Lambda_n, \alpha_1)$ is not an integer, then the polynomial solution P to equation (14) is unique up to multiplication by a nonzero number.*

Indeed, if (Λ_s, α_1) is not an integer, then P is the unique (up to multiplication by a number) solution, uni-valued in a neighborhood of z_s.

4.2. **Bethe ansatz equations for $\mathfrak{sl}_3$.** The Bethe ansatz equations (9) for $\mathfrak{sl}_3$ take the form

$$\begin{aligned} &\sum_{s=1}^{n} \frac{(\Lambda_s, \alpha_1)}{t_i^{(1)} - z_s} \;-\; \sum_{j=1,\ j\neq i}^{l_1} \frac{2}{t_i^{(1)} - t_j^{(1)}} \;+\; \sum_{j=1}^{l_2} \frac{1}{t_i^{(1)} - t_j^{(2)}} \;=\; 0\,, \qquad i = 1, \dots, l_1\,, \\ &\sum_{s=1}^{n} \frac{(\Lambda_s, \alpha_2)}{t_i^{(2)} - z_s} \;-\; \sum_{j=1,\ j\neq i}^{l_2} \frac{2}{t_i^{(2)} - t_j^{(2)}} \;+\; \sum_{j=1}^{l_1} \frac{1}{t_i^{(2)} - t_j^{1)}} \;=\; 0\,, \qquad i = 1, \dots, l_2\,. \end{aligned} \tag{15}$$

4.2.1. Introduce polynomials $F(x) = \prod_{s=1}^{n} (x - z_s)$, $P_1(x) = \prod_{i=1}^{l_1} (t_i^{(1)} - x)$, $P_2(x) = \prod_{i=1}^{l_2} (t_i^{(2)} - x)$, $F_1(x) = P_2(x)F(x)$, $F_2(x) = P_1(x)F(x)$,

$$G_1(x) \;=\; P_2'(x)F(x) \;+\; P_2(x)F(x) \sum_{s=1}^{n} \frac{(\Lambda_s, \alpha_1)}{x - z_s}\;,$$

$$G_2(x) \;=\; P_1'(x)F(x) \;+\; P_1(x)F(x) \sum_{s=1}^{n} \frac{(\Lambda_s, \alpha_2)}{x - z_s}\;.$$

We have $\deg\, F_1 = n + l_2$, $\deg\, F_2 = n + l_1$, $\deg\, G_1 = n + l_2 - 1$, $\deg\, G_2 = n + l_1 - 1$.

4.2.2. **Lemma** ([MV1]). *Assume that the roots* $t_1^{(1)}, \dots, t_{l_1}^{(1)}$, $t_1^{(2)}, \dots, t_{l_2}^{(2)}$ *of* P_1, P_2 *are all pair-wise distinct. Assume that for any* s, i, *we have* $z_s \notin \{t_1^{(i)}, \dots, t_{l_i}^{(i)}\}$.

Then the roots of P_1, P_2 *form a solution to system (15) if and only if for* $i = 1, 2$, *the polynomial* $F_i P_i'' - G_i P_i'$ *is divisible by the polynomial* P_i.

In other words, the roots of P_1, P_2 *form a solution to the Bethe ansatz equations (15) if and only if for* $i = 1, 2$, *there exists a polynomial* H_i *of degree not greater than* $\deg F_i - 2$ *such that* P_i *is a solution to the differential equation*

$$F_i \, P_i'' \; - \; G_i \, P_i' \; + \; H_i \, P_i \; = \; 0 \; . \tag{16}$$

Proof. The lemma is a corollary of the same result by Stieltjes, see [Sti] and Section 6.8 in [Sz]. We sketch the proof.

Assume that there exist such polynomials P_1, P_2, H_1, H_2. Substitute $x = t_j^{(i)}$ to equation (16). Then we get

$$\frac{P_i''(t_j^{(i)})}{P_i'(t_j^{(i)})} \; = \; \frac{G_i(t_j^{(i)})}{F_i(t_j^{(i)})} \; .$$

This is exactly the $t_j^{(i)}$-th equation in (15). Hence the roots of polynomials P_1, P_2 form a solution to equations (15). This argument is reversible. □

4.2.3. **Theorem.** *Assume that the roots* $t_1^{(1)}, \dots, t_{l_1}^{(1)}$, $t_1^{(2)}, \dots, t_{l_2}^{(2)}$ *of* P_1, P_2 *are all pair-wise distinct. Assume that for any* s, i, *we have* $z_s \notin \{t_1^{(i)}, \dots, t_{l_i}^{(i)}\}$. *Then the roots of* P_1, P_2 *form a solution to system (15) if and only if there exist numbers* $\mu_1, \dots, \mu_n$, $\mu_1 + \dots + \mu_n = 0$, *such that*

$$\begin{aligned} P_1''P_2 - P_1'P_2' + P_1P_2'' - P_1'P_2 \sum_{s=1}^n \frac{(\Lambda_s, \alpha_1)}{x - z_s} - P_1P_2' \sum_{s=1}^n \frac{(\Lambda_s, \alpha_2)}{x - z_s} + \\ + P_1P_2 \sum_{s=1}^n \frac{1}{x - z_s} \Big(\mu_s - \sum_{k \neq s} \frac{(\Lambda_s, \Lambda_k)}{z_s - z_k} \Big) \; = 0 \; . \end{aligned} \tag{17}$$

4.2.4. **Remark.** From (17) one may conclude that

$$\mu_s \; = \; \sum_{k \neq s} \frac{(\Lambda_s, \Lambda_k)}{z_s - z_k}) \; + \; (\Lambda_s, \alpha_1) \frac{P_1'}{P_1}(z_s) \; + \; (\Lambda_s, \alpha_2) \frac{P_2'}{P_2}(z_s) \; = \; \frac{\partial}{\partial z_s} \log \, \Phi(\boldsymbol{t}, \boldsymbol{z}, \boldsymbol{\Lambda}, \boldsymbol{l})) \; ,$$

c.f. Theorem 3.4.1. Thus μ_s is the eigenvalue of the s-th Gaudin operator at the Bethe vector corresponding to the solution $\boldsymbol{t}$ of the Bethe ansatz equations.

4.2.5. *Proof of Theorem 4.2.3.* Let us show that (17) implies (15). Substitute $x = t_j^{(1)}$ to (17). Then

$$\frac{P_1''(t_j^{(1)})}{P_1'(t_j^{(1)})} - \frac{P_2'(t_j^{(1)})}{P_2(t_j^{(1)})} - \sum_{s=1}^n \frac{(\Lambda_s, \alpha_1)}{t_j^{(1)} - z_s} \; = \; 0 \; .$$

This is the first of equations in (15). Substituting $x = t_j^{(2)}$ to (17) we get the second of equations in (15).

Let us show that (16) imply (17). Adding the two equations of (16) we get

$$FP_1''P_2 - FP_1'P_2' + FP_1P_2'' - FP_1'P_2\sum_{s=1}^{n}\frac{(\Lambda_s,\alpha_1)}{x-z_s} - FP_1P_2'\sum_{s=1}^{n}\frac{(\Lambda_s,\alpha_2)}{x-z_s} - \\ - FP_1'P_2' + H_1P_1 + H_2P_2 = 0\ .$$

Equation (17) will be proved if we show that $-FP_1'P_2' + H_1P_1 + H_2P_2$ is divisible by P_1P_2. For that it is enough to show that $-FP_1'P_2' + H_1P_1 + H_2P_2$ is divisible by P_1 and divisible by P_2.

From the second of equations in (16) we get

$$-FP_1'P_2' + H_2P_2 = -FP_1P_2'' + FP_1P_2'\sum_{s=1}^{n}\frac{(\Lambda_s,\alpha_2)}{x-z_s}\ .$$

Hence $-FP_1'P_2' + H_1P_1 + H_2P_2$ is divisible by P_1. Similarly it is divisible by P_2. □

4.2.6. Lemma 4.2.2 and Theorem 4.2.3 provide three ways to describe solutions to the $\mathfrak{sl}_3$ Bethe ansatz equations.

The original way: a solution is a collection $\boldsymbol{t} = (t_1^{(1)},\dots,t_{l_1}^{(1)},t_1^{(2)},\dots,t_{l_2}^{(2)})$ satisfying (15). The second way: a solution is a tuple P_1, P_2, H_1, H_2 satisfying (16). The third way: a solution is a pair P_1, P_2, and a set of numbers $\mu_1,\dots,\mu_n$, $\mu_1+\dots+\mu_n = 0$, satisfying (17).

4.2.7. There is a fourth way to describe solutions to the $\mathfrak{sl}_3$ Bethe ansatz equations. Namely, under certain conditions, the $S_{\boldsymbol{l}}$-orbits of solutions are in one to one correspondence with third order linear differential operators with regular singular points at $z_1,\dots,z_n,\infty$, with prescribed exponents at the singular points, and with a quasi-polynomial flag of solutions, see precise statements in [MTV]. Under that correspondence to a solution $\boldsymbol{t}$ one assigns the differential operator

$$D_{\boldsymbol{t}}\ =\ (\frac{d}{dx} - \ln'(\frac{T_1T_2}{P_2}))\,(\frac{d}{dx} - \ln'(\frac{P_2T_1}{P_1}))\,(\frac{d}{dx} - \ln'(P_1))\ ,$$

where $\ln'(f)$ denotes $(df/dx)/f$ for any f, and $T_i(x) = \prod_{s=1}^{n}(x-z_s)^{(\Lambda_s,\alpha_i)}$ for $i = 1,2$.

All singular points of $D_{\boldsymbol{t}}$ are regular and lie in $\{z_1,\dots,z_n,\infty\}$. The exponents of $D_{\boldsymbol{t}}$ at z_s are

$$0,(\Lambda_s,\alpha_1)+1,(\Lambda_s,\alpha_1+\alpha_2)+2,$$

for $s = 1,\dots,n$, and the exponents of $D_{\boldsymbol{t}}$ at ∞ are

$$-l_1,\ -l_1 - (\sum_{s=1}^{n}\Lambda_s - l_1\alpha_1 - l_2\alpha_2,\alpha_1) - 1,\ -l_1 - (\sum_{s=1}^{n}\Lambda_s - l_1\alpha_1 - l_2\alpha_2,\alpha_1+\alpha_2) - 2.$$

The differential equation $D_{\boldsymbol{t}}\, u = 0$ has solutions u_1, u_2, u_3 such that

$$u_1 = P_1 , \qquad \mathrm{Wr}(u_1, u_2) = P_2 T_1 , \qquad \mathrm{Wr}(u_1, u_2, u_3) = T_1^2 T_2 ,$$

where $\mathrm{Wr}\,(u_1, \dots, u_i)$ denotes the Wronskian of $u_1, \dots, u_i$.

We will not use this fourth way to describe solutions to the Bethe ansatz equations in this paper.

5. On separation of variables for $\mathfrak{sl}_2$

In this section we describe Sklyanin's separation of variables for $\mathfrak{sl}_2$ [Sk], following exposition in [St].

5.1. **Polynomial representation.** The space of polynomials $\mathbb{C}[x^{(1)}, \dots, x^{(n)}]$ is identified with the tensor product

$$M_{\boldsymbol{\Lambda}} = M_{\Lambda_1} \otimes \dots \otimes M_{\Lambda_n}$$

of $\mathfrak{sl}_2$ Verma modules by the linear map

$$(x^{(1)})^{j^1} \dots (x^{(n)})^{j^n} \mapsto e_{2,1}^{j^1} v_{\Lambda_1} \otimes \dots \otimes e_{2,1}^{j^n} v_{\Lambda_n} .$$

Then the $\mathfrak{sl}_2$ action on $\mathbb{C}[x^{(1)}, \dots, x^{(n)}]$ is given by the differential operators,

$$e^{(i)} = -x^{(i)} \partial_{x^{(i)}}^2 + (\Lambda_i, \alpha_1) \partial_{x^{(i)}} , \qquad h^{(i)} = -2x^{(i)} \partial_{x^{(i)}} + (\Lambda_i, \alpha_1) , \qquad f^{(i)} = x^{(i)} ,$$

where $\partial_{x^{(i)}}$ denotes the derivative with respect to $x^{(i)}$.

The Gaudin Hamiltonians take the form

$$H_i(\boldsymbol{z}) = \sum_{j \neq i} \frac{-x^{(i)} x^{(j)} (\partial_{x^{(i)}} - \partial_{x^{(j)}})^2 + ((\Lambda_i, \alpha_1) x^{(j)} - (\Lambda_j, \alpha_1) x^{(i)})(\partial_{x^{(i)}} - \partial_{x^{(j)}}) + (\Lambda_i, \Lambda_j)}{z_i - z_j} ,$$

$i = 1, \dots, n$.

5.2. **Change of variables.** Make the change of variables from $z_1, \dots, z_n$, $x^{(1)}, \dots, x^{(n)}$ to $z_1, \dots, z_n, u, y^{(1)}, \dots, y^{(n-1)}$ using the relation

$$\sum_{i=1}^{n} \frac{x^{(i)}}{t - z_i} = u \, \frac{\prod_{k=1}^{n-1} (t - y^{(k)})}{\prod_{i=1}^{n} (t - z_i)} , \tag{18}$$

where t is a variable. This relation defines $u, y^{(1)}, \dots, y^{(n-1)}$ uniquely up to permutation of $y^{(1)}, \dots, y^{(n-1)}$ unless $u = \sum_{i=1}^{n} x^{(i)} = 0$. The map

$$(z_1, \dots, z_n, u, y^{(1)}, \dots, y^{(n-1)}) \to (z_1, \dots, z_n, x^{(1)}, \dots, x^{(n)})$$

is an unramified covering on the complement to the union of diagonals $y^{(i)} = y^{(j)}$ and the hyperplane $u = 0$.

5.2.1. We have

$$x^{(i)} = u \, \frac{\prod_{j=1}^{n-1} (z_i - y^{(j)})}{\prod_{s\neq i} (z_i - z_s)} \,, \qquad \frac{\partial y^{(j)}}{\partial x^{(i)}} = - \, \frac{\prod_{s\neq i} (y^{(j)} - z_s)}{u \, \prod_{l\neq j} (y^{(j)} - y^{(l)})}$$

and then

$$\partial_{x^{(i)}} = \partial_u - \frac{1}{u} \sum_{j=1}^{n-1} \frac{\prod_{s\neq i} (y^{(j)} - z_s)}{\prod_{l\neq j} (y^{(j)} - y^{(l)})} \, \partial_{y^{(j)}} \,,$$

$$\partial_{x^{(i)}} - \partial_{x^{(j)}} = \frac{z_j - z_i}{u} \sum_{l=1}^{n-1} \frac{\prod_{s\notin\{i,j\}} (y^{(l)} - z_s)}{\prod_{m\neq l} (y^{(l)} - y^{(m)})} \, \partial_{y^{(l)}} \,.$$

5.3. **Eigenvectors of the Gaudin Hamiltonians.** Assume that we have a common eigenfunction of the Gaudin Hamiltonians $H_i(\boldsymbol{z})$ with eigenvalues μ_i. Then the eigenfunction is annihilated by the operators $H_i(\boldsymbol{z}) - \mu_i$.

Recall that $H_1(\boldsymbol{z}) + \cdots + H_n(\boldsymbol{z}) = 0$ and hence $\mu_1 + \cdots + \mu_n = 0$.

Consider the following operators:

$$K_j(\boldsymbol{z}) = \sum_{i=1}^{n} \frac{1}{y^{(j)} - z_i} \, (H_i(\boldsymbol{z}) - \mu_i) \,, \qquad j = 1, \ldots, n-1 \,.$$

They annihilate a common eigenfunction of the Gaudin Hamiltonians $H_i(\boldsymbol{z})$ with eigenvalues μ_i.

5.3.1. **Theorem** ([Sk], [St]). *In variables $u, y^{(1)}, \ldots, y^{(n-1)}$, we have*

$$K_j(\boldsymbol{z}) = -\partial^2_{y^{(j)}} + \sum_{i=1}^{n} \frac{(\Lambda_i, \alpha_1)}{y^{(j)} - z_i} \, \partial_{y^{(j)}} + \sum_{i=1}^{n} \frac{1}{y^{(j)} - z_i} \, (-\mu_i + \sum_{k\neq i} \frac{(\Lambda_i, \Lambda_k)}{z_i - z_k}) \,.$$

This is the main point of the separation of variables: the operator $K_j(\boldsymbol{z})$ depends only on $y^{(j)}$ and does not depend on other variables $y^{(j')}$ and u; this differential operator is the same operator for all j; moreover, it is the same operator as in the Bethe ansatz equation (14).

5.3.2. More precisely, Theorem 5.3.1 claims two identities:

$$\partial^2_{y^{(j)}} = \sum_{i=1}^{n} \sum_{k\neq i} \frac{x^{(i)} x^{(k)}}{(y^{(j)} - z_i)(z_i - z_k)} \, (\partial_{x^{(i)}} - \partial_{x^{(k)}})^2 \,, \tag{19}$$

$$\sum_{i=1}^{n} \frac{(\Lambda_i, \alpha_1)}{y^{(j)} - z_i} \, \partial_{y^{(j)}} = \sum_{i=1}^{n} \sum_{k\neq i} \frac{(\Lambda_i, \alpha_1) x^{(k)} - (\Lambda_k, \alpha_1) x^{(i)}}{(y^{(j)} - z_i)(z_i - z_k)} \, (\partial_{x^{(i)}} - \partial_{x^{(k)}}) \,. \tag{20}$$

5.4. **Canonical weight function.** Fix a weight subspace $M_{\boldsymbol{\Lambda}}[\boldsymbol{l}] \subset M_{\boldsymbol{\Lambda}}$, $\boldsymbol{l} = (l_1)$. This weight subspace corresponds to the subspace of $\mathbb{C}[x^{(1)}, \dots, x^{(n)}]$ of homogeneous polynomials of degree l_1. The canonical weight function $\omega_{\boldsymbol{l},n}$ becomes the following function of $x^{(1)}, \dots, x^{(n)}$, $z_1, \dots, z_n$, $t_1^{(1)}, \dots, t_{l_1}^{(1)}$,

$$\prod_{j=1}^{l_1} \Big(\sum_{i=1}^{n} \frac{x^{(i)}}{t_j^{(1)} - z_i} \Big) \,.$$

In variables $u, y^{(1)}, \dots, y^{(n-1)}$, $z_1, \dots, z_n$, $t_1^{(1)}, \dots, t_{l_1}^{(1)}$, the canonical weight function is

$$\Psi(\boldsymbol{t}, \boldsymbol{z}, u, \boldsymbol{y}) \;=\; u^{l_1} \, \frac{P(y^{(1)}) \dots P(y^{(n-1)})}{P(z_1) \dots P(z_n)} \;, \tag{21}$$

where $P(x) = \prod_{i=1}^{l_1} (t_i^{(1)} - x)$ as in Section 4.1.

5.4.1. Notice that the canonical weight function, as a function of u, $y^{(1)}$, …, $y^{(n-1)}$, is the product of functions of one variable. This is another manifestation of separation of variables.

5.4.2. **Theorem.** *Assume that the numbers $\boldsymbol{z} = (z_1, \dots, z_n)$ are distinct and the numbers $\boldsymbol{t} = (t_1^{(1)}, \dots, t_{l_1}^{(1)})$ are distinct. Assume that for any s we have $z_s \notin \{t_1^{(1)}, \dots, t_{l_1}^{(1)}\}$. Assume that for such a $\boldsymbol{t}$, the canonical weight function $\Psi(\boldsymbol{t}, \boldsymbol{z}, u, \boldsymbol{y})$, as a function of $u, \boldsymbol{y}$, is an eigenvector of the Gaudin Hamiltonians. Then $\boldsymbol{t}$ is a solution to the Bethe ansatz equations (12).*

Proof. If $\Psi(\boldsymbol{t}, \boldsymbol{z}, u, \boldsymbol{y})$ is an eigenfunction, then it is annihilated by the operators $K_j(\boldsymbol{z})$, $j = 1, \dots, n-1$. Hence $\boldsymbol{t}$ is a solution of (12) by Corollary 4.1.4. □

5.4.3. Theorem 3.4.1 says that if $\boldsymbol{t}$ is a solution to the Bethe ansatz equations, then the value at $\boldsymbol{t}$ of the canonical weight function is an eigenvector of the Gaudin Hamiltonians and is a singular vector. Theorem 5.4.2 gives a converse statement: if the value of the canonical weight function at some point $\boldsymbol{t}$ is an eigenvector of the Gaudin Hamiltonians, then $\boldsymbol{t}$ is a solution to the Bethe ansatz equations and that value is a singular vector.

5.5. **Lemma.** *Let at least one of the numbers $(\Lambda_1, \alpha_1), \dots, (\Lambda_n, \alpha_1)$ be not an integer. Let $f(y^{(1)}, \dots, y^{(n-1)})$ be a polynomial. Assume that for some $\mu_1, \dots, \mu_n$, $\mu_1 + \dots + \mu_n = 0$, we have $K_i(z) f = 0$ for $i = 1, \dots, n-1$. Then there exists a polynomial P of one variable such that*

$$f(y^{(1)}, \dots, y^{(n-1)}) \;=\; P(y^{(1)}) \dots P(y^{(n-1)}) \,.$$

Proof. Equation $K_1(\boldsymbol{z}) f = 0$ implies that $f(y^{(1)}, \dots, y^{(n-1)}) = P(y^{(1)})\, g(y^{(2)}, \dots, y^{(n-1)})$, where $P(y^{(1)})$ is a polynomial and $g(y^{(1)}, \dots, y^{(n-1)})$ a suitable function. The polynomial P is unique up to multiplication by a number. Applying the same reasoning to g we get the lemma. □

6. On separation of variables for $\mathfrak{sl}_3$

6.1. **Polynomial representation.** The space of polynomials $\mathbb{C}[x_k^{(i)}]_{k=1,2,3}^{i=1,\dots,n}$ of $3n$ variables is identified with the tensor product

$$M_{\mathbf{\Lambda}} \;=\; M_{\Lambda_1}\otimes\cdots\otimes M_{\Lambda_n}$$

of $\mathfrak{sl}_3$ Verma modules by the linear map

$$\begin{aligned}(x_1^{(1)})^{j_1^1}\,(x_3^{(1)})^{j_3^1}\,(x_2^{(1)})^{j_2^1}\,\dots\,(x_1^{(n)})^{j_1^n}\,(x_3^{(n)})^{j_3^n}\,(x_2^{(n)})^{j_2^n}&\\ \mapsto\; e_{2,1}^{j_1^1}\,e_{3,1}^{j_3^1}\,e_{3,2}^{j_2^1}\,v_{\Lambda_1}\otimes\;\dots\;\otimes e_{2,1}^{j_1^n}\,e_{3,1}^{j_3^n}\,e_{3,2}^{j_2^n}\,v_{\Lambda_n}\;.&\end{aligned}$$

Then the $\mathfrak{sl}_3$ action on $\mathbb{C}[x_j^{(i)}]$ is given by the differential operators,

$$\begin{aligned}
e_{2,1}^{(i)} &= \quad x_1^{(i)}\;, \qquad e_{3,2}^{(i)} = x_2^{(i)} + x_3^{(i)}\partial_{x_1^{(i)}}\;, \qquad e_{3,1}^{(i)} = x_3^{(i)}\;,\\
h_1^{(i)} &= -\;\; 2x_1^{(i)}\partial_{x_1^{(i)}} + x_2^{(i)}\partial_{x_2^{(i)}} - x_3^{(i)}\partial_{x_3^{(i)}} + (\Lambda_i,\alpha_1)\;,\\
h_2^{(i)} &= -\;\; 2x_2^{(i)}\partial_{x_2^{(i)}} + x_1^{(i)}\partial_{x_1^{(i)}} - x_3^{(i)}\partial_{x_3^{(i)}} + (\Lambda_i,\alpha_2)\;,\\
e_{1,2}^{(i)} &= -\;\; x_1^{(i)}\partial^2_{x_1^{(i)}} + x_2^{(i)}\partial_{x_1^{(i)}}\partial_{x_2^{(i)}} - x_2^{(i)}\partial_{x_3^{(i)}} - x_3^{(i)}\partial_{x_3^{(i)}}\partial_{x_1^{(i)}} + (\Lambda_i,\alpha_1)\partial_{x_1^{(i)}}\;,\\
e_{2,3}^{(i)} &= -\;\; x_2^{(i)}\partial^2_{x_2^{(i)}} + x_1^{(i)}\partial_{x_3^{(i)}} + (\Lambda_i,\alpha_2)\partial_{x_2^{(i)}}\;.\\
e_{1,3}^{(i)} &= -\;\; x_3^{(i)}\partial^2_{x_3^{(i)}} - x_1^{(i)}\partial_{x_1^{(i)}}\partial_{x_3^{(i)}} + x_2^{(i)}\partial_{x_1^{(i)}}\partial^2_{x_2^{(i)}} - x_2^{(i)}\partial_{x_2^{(i)}}\partial_{x_3^{(i)}} -\\
&\quad -\;\; (\Lambda_i,\alpha_2)\partial_{x_1^{(i)}}\partial_{x_2^{(i)}} + (\Lambda_i,\alpha_1+\alpha_2)\partial_{x_3^{(i)}}\;.
\end{aligned}$$

Then we have the following formula for the Casimir operator:

$$\begin{aligned}
\Omega^{(i,j)} &= \big\{-x_1^{(i)}\partial^2_{x_1^{(i)}} + x_2^{(i)}(\partial_{x_1^{(i)}}\partial_{x_2^{(i)}} - \partial_{x_3^{(i)}}) - x_3^{(i)}\partial_{x_1^{(i)}}\partial_{x_3^{(i)}} + (\Lambda_i,\alpha_1)\partial_{x_1^{(i)}}\big\}x_1^{(j)} +\\
&+\quad x_1^{(i)}\big\{-x_1^{(j)}\partial^2_{x_1^{(j)}} + x_2^{(j)}(\partial_{x_1^{(j)}}\partial_{x_2^{(j)}} - \partial_{x_3^{(j)}}) - x_3^{(j)}\partial_{x_1^{(j)}}\partial_{x_3^{(j)}} + (\Lambda_j,\alpha_1)\partial_{x_1^{(j)}}\big\} +\\
&+\quad \big\{-x_2^{(i)}\partial^2_{x_2^{(i)}} + x_1^{(i)}\partial_{x_3^{(i)}} + (\Lambda_i,\alpha_2)\partial_{x_2^{(i)}}\big\}\big\{x_2^{(j)} + x_3^{(j)}\partial_{x_1^{(j)}}\big\} +\\
&+\quad \big\{x_2^{(i)} + x_3^{(i)}\partial_{x_1^{(i)}}\big\}\big\{-x_2^{(j)}\partial^2_{x_2^{(j)}} + x_1^{(j)}\partial_{x_3^{(j)}} + (\Lambda_j,\alpha_2)\partial_{x_2^{(j)}}\big\} +\\
&+\quad \big\{-x_3^{(i)}\partial^2_{x_3^{(i)}} - x_1^{(i)}\partial_{x_1^{(i)}}\partial_{x_3^{(i)}} + x_2^{(i)}\partial_{x_1^{(i)}}\partial^2_{x_2^{(i)}} - x_2^{(i)}\partial_{x_2^{(i)}}\partial_{x_3^{(i)}} -\\
&\qquad\qquad - (\Lambda_i,\alpha_2)\partial_{x_1^{(i)}}\partial_{x_2^{(i)}} + (\Lambda_i,\alpha_1+\alpha_2)\partial_{x_3^{(i)}}\big\}x_3^{(j)} +\\
&+\quad x_3^{(i)}\big\{-x_3^{(j)}\partial^2_{x_3^{(j)}} - x_1^{(j)}\partial_{x_1^{(j)}}\partial_{x_3^{(j)}} + x_2^{(j)}\partial_{x_1^{(j)}}\partial^2_{x_2^{(j)}} - x_2^{(j)}\partial_{x_2^{(j)}}\partial_{x_3^{(j)}} -\\
&\qquad\qquad - (\Lambda_j,\alpha_2)\partial_{x_1^{(j)}}\partial_{x_2^{(j)}} + (\Lambda_j,\alpha_1+\alpha_2)\partial_{x_3^{(j)}}\big\} +
\end{aligned}$$

$$+\big\{-2x_1^{(i)}\partial_{x_1^{(i)}} + x_2^{(i)}\partial_{x_2^{(i)}} - x_3^{(i)}\partial_{x_3^{(i)}} + (\Lambda_i,\alpha_1)\big\}\big\{-x_1^{(j)}\partial_{x_1^{(j)}} - x_3^{(j)}\partial_{x_3^{(j)}} + (\Lambda_j,\frac{2\alpha_1+\alpha_2}{3})\big\} +$$

$$+\big\{-2x_2^{(i)}\partial_{x_2^{(i)}} + x_1^{(i)}\partial_{x_1^{(i)}} - x_3^{(i)}\partial_{x_3^{(i)}} + (\Lambda_i,\alpha_2)\big\}\big\{-x_2^{(j)}\partial_{x_2^{(j)}} - x_3^{(j)}\partial_{x_3^{(j)}} + (\Lambda_j, \frac{\alpha_1+2\alpha_2}{3})\big\}\ .$$

Rearranging the terms we get

$$\Omega^{(i,j)} \ = \ \Omega_0^{(i,j)} + \tilde\Omega_0^{(i,j)} + \Omega_{>0}^{(i,j)} + \Omega_{<0}^{(i,j)}\ , \tag{22}$$

where

$$\begin{aligned}\Omega_0^{(i,j)} \ &= \ (\Lambda_i,\Lambda_j) + \big(x_1^{(j)}x_2^{(i)}\partial_{x_2^{(i)}} - x_1^{(i)}x_2^{(j)}\partial_{x_2^{(j)}}\big)\big(\partial_{x_1^{(i)}} - \partial_{x_1^{(j)}}\big) \ - \\ &- \ x_1^{(i)}x_1^{(j)}\big(\partial_{x_1^{(i)}} - \partial_{x_1^{(j)}}\big)^2 - x_2^{(i)}x_2^{(j)}\big(\partial_{x_2^{(i)}} - \partial_{x_2^{(j)}}\big)^2 \ + \\ &+ \ \big((\Lambda_i,\alpha_1)x_1^{(j)} - (\Lambda_j,\alpha_1)x_1^{(i)}\big)\big(\partial_{x_1^{(i)}} - \partial_{x_1^{(i)}}\big) \ + \\ &+ \ \big((\Lambda_i,\alpha_2)x_2^{(j)} - (\Lambda_j,\alpha_2)x_2^{(i)}\big)\big(\partial_{x_2^{(i)}} - \partial_{x_2^{(j)}}\big)\ ,\end{aligned} \tag{23}$$

$$\begin{aligned}\tilde\Omega_0^{(i,j)} \ &= \ -\big(x_3^{(i)}x_1^{(j)} + x_1^{(i)}x_3^{(j)}\big)\big(\partial_{x_1^{(i)}} - \partial_{x_1^{(j)}}\big)\big(\partial_{x_3^{(i)}} - \partial_{x_3^{(j)}}\big) - \\ &- \ x_3^{(i)}x_3^{(j)}\big(\partial_{x_3^{(i)}} - \partial_{x_3^{(j)}}\big)^2 + \big(x_2^{(j)}x_3^{(i)}\partial_{x_2^{(j)}} - x_2^{(i)}x_3^{(j)}\partial_{x_2^{(i)}}\big)\big(\partial_{x_3^{(i)}} - \partial_{x_3^{(j)}}\big) \ + \\ &+ \ \big((\Lambda_i,\alpha_1+\alpha_2)x_3^{(j)} \ - (\Lambda_j,\alpha_1+\alpha_2)x_3^{(i)}\big)\big(\partial_{x_3^{(i)}} - \partial_{x_3^{(j)}}\big)\ ,\end{aligned} \tag{24}$$

$$\begin{aligned}\Omega_{>0}^{(i,j)} \ &= \ \big(x_2^{(i)}x_3^{(j)}\partial^2_{x_2^{(i)}} - x_3^{(i)}x_2^{(j)}\partial^2_{x_2^{(j)}}\big)\big(\partial_{x_1^{(i)}} - \partial_{x_1^{(j)}}\big) \ + \\ &+ \ \big((\Lambda_j,\alpha_2)x_3^{(i)}\partial_{x_2^{(j)}} - (\Lambda_i,\alpha_2)x_3^{(j)}\partial_{x_2^{(i)}}\big)\big(\partial_{x_1^{(i)}} - \partial_{x_1^{(j)}}\big)\ ,\end{aligned} \tag{25}$$

$$\Omega_{<0}^{(i,j)} \ = \ \big(x_1^{(i)}x_2^{(j)} - x_1^{(j)}x_2^{(i)}\big)\big(\partial_{x_3^{(i)}} - \partial_{x_3^{(j)}}\big)\ . \tag{26}$$

The meaning of this decomposition of the Casimir element will be explained in Section 6.4.

6.2. **Change of variables.** Make the change of variables from $z_1,\dots,z_n$, $x_1^{(1)},x_2^{(1)},x_3^{(1)}$, $\dots$, $x_1^{(n)},x_2^{(n)},x_3^{(n)}$ to $z_1,\dots,z_n$, u_1,u_2,u_3, $y_1^{(1)},y_2^{(1)},y_3^{(1)},\dots,y_1^{(n-1)},y_2^{(n-1)},y_3^{(n-1)}$ using the relations

$$\sum_{i=1}^n \frac{x_k^{(i)}}{t-z_i} \ = \ u_k\,\frac{\prod_{j=1}^{n-1}(t-y_k^{(j)})}{\prod_{i=1}^n(t-z_i)}\ , \qquad k=1,2,3\ , \tag{27}$$

where t is a variable.

Denote $\boldsymbol{x} = (x_1^{(1)},x_2^{(1)},x_3^{(1)},\ \dots,\ x_1^{(n)},x_2^{(n)},x_3^{(n)})$, $\boldsymbol{u} = (u_1,u_2,u_3)$, $\boldsymbol{y} = (y_1^{(1)},\ y_2^{(1)},\ y_3^{(1)}$, $\dots,\ y_1^{(n-1)},y_2^{(n-1)},y_3^{(n-1)})$.

Relations (27) define $(\boldsymbol{u},\boldsymbol{y})$ uniquely up to a permutation of $y_k^{(j)}$'s, which preserves the lower index, unless $u_k = \sum_{i=1}^n x_k^{(i)} = 0$ for some of k's. The map

$$(\boldsymbol{z},\boldsymbol{u},\boldsymbol{y}) \ \to \ (\boldsymbol{z},\boldsymbol{x})$$

is an unramified covering on the complement to the union of diagonals $y_k^{(i)} = y_k^{(j)}$ and the hyperplanes $u_k = 0$.

6.3. **Degree on $M_{\mathbf{\Lambda}}[\boldsymbol{l}]$.**

6.3.1. For an index $J = (j_1^1, j_3^1, j_2^1 \ldots, j_1^n, j_3^n, j_2^n)$ and $k = 1,2,3$, set $J_k = j_k^1 + \cdots + j_k^n$.
A monomial

$$X_J \;=\; (x_1^{(1)})^{j_1^1}\,(x_3^{(1)})^{j_3^1}\,(x_2^{(1)})^{j_2^1}\,\ldots\,(x_1^{(n)})^{j_1^n}\,(x_3^{(n)})^{j_3^n}\,(x_2^{(n)})^{j_2^n}$$

belongs to a weight subspace $M_{\mathbf{\Lambda}}[\boldsymbol{l}] \subset M_{\mathbf{\Lambda}}$, $\boldsymbol{l} = (l_1, l_2)$, if $J_1 + J_3 = l_1$ and $J_3 + J_2 = l_2$.
We will consider the decomposition

$$M_{\mathbf{\Lambda}}[\boldsymbol{l}] \;=\; \oplus_{d=0}^{\min(l_1,l_2)} M_{\mathbf{\Lambda},d}[\boldsymbol{l}] \;,$$

where $M_{\mathbf{\Lambda},d}[\boldsymbol{l}]$ is spanned by all monomials X_J with $J_1 = l_1 - d$, $J_3 = d$, $J_2 = l_2 - d$. We say that $M_{\mathbf{\Lambda},d}[\boldsymbol{l}]$ consists of *elements of degree d*.

6.3.2. In coordinates $(\boldsymbol{u}, \boldsymbol{y})$, an element belongs to $M_{\mathbf{\Lambda},d}[\boldsymbol{l}]$, i.e. has degree d, if it has the form

$$u_1^{l_1-d}\,u_3^{d}\,u_2^{l_2-d}\,f_d \;,$$

where f_d is a polynomial in $\boldsymbol{y}$. This polynomial does not depend on $y_3^{(1)}, \ldots, y_3^{(n-1)}$ if $d=0$. An arbitrary element of $M_{\mathbf{\Lambda}}[\boldsymbol{l}]$ has degree decomposition:

$$\begin{aligned} F \;=\; & u_1^{l_1}\,u_2^{l_2}\,f_0(y_1^{(1)}, y_2^{(1)}, \ldots, y_1^{(n-1)}, y_2^{(n-1)}) \;+ \\ & + \sum_{d=1}^{\min(l_1,l_2)} u_1^{l_1-d}\,u_3^{d}\,u_2^{l_2-d}\,f_d(y_1^{(1)}, y_3^{(1)}, y_2^{(1)}, \ldots, y_1^{(n-1)}, y_3^{(n-1)}, y_2^{(n-1)}) \;. \end{aligned}$$

6.4. **Casimir element and degree.** The Casimir element $\Omega^{(i,j)}$ is given in (22) – (26). Using formulae of Sections 6.2, consider the Casimir element as an operator acting on functions of $(\boldsymbol{u}, \boldsymbol{y})$.

6.4.1. **Lemma.**

- *The operator $\Omega_0^{(i,j)}$, given in (23), preserves the degree introduced in Section 6.3 and does not contain derivatives with respect to variables $y_3^{(1)}, \ldots, y_3^{(n-1)}$.*
- *The operator $\tilde{\Omega}_0^{(i,j)}$, given in (24), preserves the degree and annihilates functions which do not depend on variables $y_3^{(1)}, \ldots, y_3^{(n-1)}$.*
- *The operator $\Omega_{>0}^{(i,j)}$, given in (25), increases the degree by one.*
- *The operator $\Omega_{<0}^{(i,j)}$, given in (26), decreases the degree by one and annihilates functions which do not depend on variables $y_3^{(1)}, \ldots, y_3^{(n-1)}$.*

The proof is evident.

6.4.2. In the $3n$-dimensional space with coordinates $(\boldsymbol{u}, \boldsymbol{y})$ consider the subspace $\mathfrak{D}$ defined by equations

$$y_1^{(j)} \;=\; y_2^{(j)}\;, \qquad j = 1, \dots, n-1\;.$$

The subspace will be called *the main diagonal.*

6.4.3. **Lemma.** *Let F be a polynomial in $(\boldsymbol{u}, \boldsymbol{y})$. Apply $\Omega^{(i,j)}_{<0}$ to F. Then the restriction of the function $\Omega^{(i,j)}_{<0} F$ to the main diagonal equals zero.*

Indeed, the factor $\big(x_1^{(i)} x_2^{(j)} - x_1^{(j)} x_2^{(i)}\big)$ in $\Omega^{(i,j)}_{<0}$ is zero on $\mathfrak{D}$.

6.4.4. Let F be an element in $M_{\boldsymbol{\Lambda}}[\boldsymbol{l}]$. Let $F = F_0 + F_1 + \dots$ and $\Omega^{(i,j)} F = (\Omega^{(i,j)} F)_0 + (\Omega^{(i,j)} F)_1 + \dots$ be the degree decompositions.

Lemma. *The restrictions to $\mathfrak{D}$ of the functions $(\Omega^{(i,j)} F)_0$ and $\Omega^{(i,j)}_0 F_0$ coincide,*

$$(\Omega^{(i,j)} F)_0|_{\mathfrak{D}} \;=\; (\Omega^{(i,j)}_0 F_0)|_{\mathfrak{D}}\;.$$

In other words, the restriction to $\mathfrak{D}$ of the leading term $(\Omega^{(i,j)} F)_0$ can be calculated using only the operator $\Omega^{(i,j)}_0$ applied to F_0 and then restricted to $\mathfrak{D}$.

Proof. The polynomial F_0 does not depend on $y_3^{(1)}, \dots, y_3^{(n-1)}$. Hence $\tilde\Omega^{(i,j)}_0 F_0 = 0$ and $\Omega^{(i,j)}_{<0} F_0 = 0$. The function $\Omega^{(i,j)}_{>0} F_0$ has degree one. The function $\Omega^{(i,j)}(F_2 + F_3 + \dots)$ has no degree zero part. The restriction to $\mathfrak{D}$ of the degree zero part of the function $\Omega^{(i,j)} F_1$ is zero by Lemma 6.4.3. This proves the lemma. □

6.5. **Eigenvectors of the Gaudin Hamiltonians.**

6.5.1. Assume that we have a common eigenfunction F of the Gaudin Hamiltonians $H_i(\boldsymbol{z})$ with eigenvalues μ_i, $i = 1, \dots, n$. Then the eigenfunction is annihilated by the operators $H_i(\boldsymbol{z}) - \mu_i$. Recall that $\mu_1 + \dots + \mu_n = 0$.

Consider the following operators:

$$K_j(\boldsymbol{z}) \;=\; \sum_{i=1}^{n} \frac{1}{y^{(j)} - z_i}\,(H_i(\boldsymbol{z}) - \mu_i)\;, \qquad j = 1, \dots, n-1\;.$$

They annihilate a common eigenfunction of the Gaudin Hamiltonians $H_i(\boldsymbol{z})$ with eigenvalues μ_i.

6.5.2. Decomposition (22) of the Casimir operators into graded components, induces the decomposition of the Gaudin Hamiltonians into graded components,

$$H_i(\boldsymbol{z}) = \ H_i(\boldsymbol{z})_0 + \tilde{H}_i(\boldsymbol{z})_0 + H_i(\boldsymbol{z})_{>0} + H_i(\boldsymbol{z})_{<0} \ ,$$

and the decomposition into graded components

$$K_j(\boldsymbol{z}) = \ K_j(\boldsymbol{z})_0 + \tilde{K}_j(\boldsymbol{z})_0 + K_j(\boldsymbol{z})_{>0} + K_j(\boldsymbol{z})_{<0} \ ,$$

where

$$K_j(\boldsymbol{z})_0 \ = \ \sum_{i=1}^n \frac{1}{y_1^{(j)} - z_i} \ (H_i(\boldsymbol{z})_0 - \mu_i) \ , \qquad \tilde{K}_j(\boldsymbol{z})_0 \ = \ \sum_{i=1}^n \frac{1}{y_1^{(j)} - z_i} \ \tilde{H}_i(\boldsymbol{z})_0 \ ,$$

$$K_j(\boldsymbol{z})_{>0} \ = \ \sum_{i=1}^n \frac{1}{y_1^{(j)} - z_i} \ H_i(\boldsymbol{z})_{>0} \ , \qquad K_j(\boldsymbol{z})_{<0} \ = \ \sum_{i=1}^n \frac{1}{y_1^{(j)} - z_i} \ H_i(\boldsymbol{z})_{<0} \ .$$

6.5.3. **Lemma.** *Let $F = F_0 + F_1 + \dots$ be the degree decomposition of the common eigenfunction of the Gaudin Hamiltonians, then we have*

$$(K_j(\boldsymbol{z})_0 F_0)|_{\mathfrak{D}} \ = \ 0 \ , \qquad j = 1, \dots, n-1 \ . \tag{28}$$

The lemma is a corollary of Lemma 6.4.4.

These are important equations. Later, under certain conditions, we will show how to find F_0 from these equations and how to recover F knowing F_0, see Theorem 6.7.4.

6.5.4. Let F be the common eigenfunction of the Gaudin Hamiltonians, $F \in M_{\boldsymbol{\Lambda}}[\boldsymbol{l}]$, $\boldsymbol{l} = (l_1, l_2)$. Then

$$F_0 \ = \ u_1^{l_1} u_2^{l_2} f \ ,$$

where f is a polynomial in $y_1^{(j)}, y_2^{(j)}$, $j = 1, \dots, n-1$, see Section 6.3.

Theorem. *Equations (28) have the form*

$$\begin{aligned}\Big(- \frac{\partial^2 f}{\partial y_1^{(j)} \partial y_1^{(j)}} \ + \ \frac{\partial^2 f}{\partial y_1^{(j)} \partial y_2^{(j)}} \ - \ \frac{\partial^2 f}{\partial y_2^{(j)} \partial y_2^{(j)}} \ - \ \frac{\partial f}{\partial y_1^{(j)}} \sum_{i=1}^n \frac{(\Lambda_i, \alpha_1)}{y_1^{(j)} - z_i} \ - \\ - \ \frac{\partial f}{\partial y_2^{(j)}} \sum_{i=1}^n \frac{(\Lambda_i, \alpha_2)}{y_1^{(j)} - z_i} \ + \ f \sum_{i=1}^n \frac{1}{y_1^{(j)} - z_i} \, (-\mu_i + \sum_{k \neq i} \frac{(\Lambda_i, \Lambda_k)}{z_i - z_k} \,) \Big) \Big|_{\mathfrak{D}} \ = \ 0 \ .\end{aligned} \tag{29}$$

Recall that $\mathfrak{D}$ is defined by the conditions $y_1^{(j)} = y_2^{(j)}$ for $j = 1, \dots, n-1$.

6.5.5. As in Sklyanin's Theorem 5.3.1, equations (29) of Theorem 6.5.4 have three interesting properties.

The first property is that the variables have separated. Namely, the j-th equation $(K_j(\boldsymbol{z})_0 F_0)|_{\mathfrak{D}} = 0$ depends only on variables $y_1^{(j)}, y_2^{(j)}$ (which at the end are put being equal) and does not depend on other variables $y_1^{(j')}$, $y_2^{(j')}$ and u_1, u_2.

The second property is that the differential operator in (29) is the same equation for all indices j.

The third property is that the operator in (29) is the same as the operator in the Bethe ansatz equation (17) of Theorem 4.2.3.

6.5.6. *Proof of Theorem 6.5.4.* The theorem follows from Theorem 5.3.1. Indeed, in order to prove Theorem 6.5.4 it is enough to prove five identities:

$$\frac{\partial^2 F_0}{\partial y_1^{(j)} \partial y_1^{(j)}}\bigg|_{\mathfrak{D}} = \Big(\sum_{i=1}^{n}\sum_{k\neq i} \frac{x_1^{(i)} x_1^{(k)}}{(y_1^{(j)} - z_i)(z_i - z_k)} (\partial_{x_1^{(i)}} - \partial_{x_1^{(k)}})^2 F_0\Big)\bigg|_{\mathfrak{D}} ,$$

$$\frac{\partial^2 F_0}{\partial y_2^{(j)} \partial y_2^{(j)}}\bigg|_{\mathfrak{D}} = \Big(\sum_{i=1}^{n}\sum_{k\neq i} \frac{x_2^{(i)} x_2^{(k)}}{(y_1^{(j)} - z_i)(z_i - z_k)} (\partial_{x_2^{(i)}} - \partial_{x_2^{(k)}})^2 F_0\Big)\bigg|_{\mathfrak{D}} ,$$

$$\frac{\partial^2 F_0}{\partial y_1^{(j)} \partial y_2^{(j)}}\bigg|_{\mathfrak{D}} =$$

$$= \Big(\sum_{i=1}^{n}\sum_{k\neq i} \frac{1}{(y_1^{(j)} - z_i)(z_i - z_k)} \big(x_1^{(k)} x_2^{(i)} \partial_{x_2^{(i)}} - x_1^{(i)} x_2^{(k)} \partial_{x_2^{(k)}}\big)\big(\partial_{x_1^{(i)}} - \partial_{x_1^{(k)}}\big) F_0\Big)\bigg|_{\mathfrak{D}} ,$$

$$\frac{\partial F_0}{\partial y_1^{(j)}} \sum_{i=1}^{n} \frac{(\Lambda_i, \alpha_1)}{y_1^{(j)} - z_i}\bigg|_{\mathfrak{D}} = \Big(\sum_{i=1}^{n}\sum_{k\neq i} \frac{(\Lambda_i, \alpha_1) x_1^{(k)} - (\Lambda_k, \alpha_1) x_1^{(i)}}{(y_1^{(j)} - z_i)(z_i - z_k)} (\partial_{x_1^{(i)}} - \partial_{x_1^{(k)}}) F_0\Big)\bigg|_{\mathfrak{D}} ,$$

$$\frac{\partial F_0}{\partial y_2^{(j)}} \sum_{i=1}^{n} \frac{(\Lambda_i, \alpha_2)}{y_1^{(j)} - z_i}\bigg|_{\mathfrak{D}} = \Big(\sum_{i=1}^{n}\sum_{k\neq i} \frac{(\Lambda_i, \alpha_2) x_2^{(k)} - (\Lambda_k, \alpha_2) x_2^{(i)}}{(y_1^{(j)} - z_i)(z_i - z_k)} (\partial_{x_2^{(i)}} - \partial_{x_2^{(k)}}) F_0\Big)\bigg|_{\mathfrak{D}} .$$

But the first three identities are corollaries of identity (19), and the last two identities are corollaries of identity (20). □

6.6. **Canonical weight function.** Fix a weight subspace $M_{\boldsymbol{\Lambda}}[\boldsymbol{l}] \subset M_{\boldsymbol{\Lambda}}$, $\boldsymbol{l} = (l_1, l_2)$. In the polynomial representation, the canonical weight function $\omega_{\boldsymbol{l},n}$ becomes a function in $x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, \dots, x_1^{(n)}, x_2^{(n)}, x_3^{(n)}, z_1, \dots, z_n, t_1^{(1)}, \dots, t_{l_1}^{(1)}, t_1^{(2)}, \dots, t_{l_2}^{(2)}$. After the change of variables of Section 6.2 it becomes a function in u_1, u_2, u_3, $y_1^{(1)}, y_2^{(1)}, y_3^{(1)} \dots$, $y_1^{(n-1)}, y_2^{(n-1)}, y_3^{(n-1)}$, $z_1, \dots, z_n$, $t_1^{(1)}, \dots, t_{l_1}^{(1)}$, $t_1^{(2)}, \dots, t_{l_2}^{(2)}$.

We will give a formula for the canonical weight function and its degree decomposition. First we prepare notation.

6.6.1. For a given $d = 1, \dots, \min(l_1, l_2)$, we will sum certain terms over pairs of ordered subsets $(\boldsymbol{k}, \boldsymbol{m})$, such that $\boldsymbol{k} = (k_1, \dots, k_d)$, $\boldsymbol{m} = (m_1, \dots, m_d)$, where $k_1, \dots, k_d$ are distinct elements of $(1, \dots, l_1)$ and $m_1, \dots, m_d$ are distinct elements of $(1, \dots, l_2)$ Such $(\boldsymbol{k}, \boldsymbol{m})$ will be called *d-admissible*. The summation over d-admissible pairs will be denoted $\sum_{d-\mathrm{adm}\,(\boldsymbol{k},\boldsymbol{m})}$.

To a d-admissible pair $(\boldsymbol{k}, \boldsymbol{m})$, we assign the following function

$$\Xi_{(\boldsymbol{k},\boldsymbol{m})}(\boldsymbol{t}, \boldsymbol{z}, \boldsymbol{y}) \ = \ = \prod_{a=1}^{d} \Big(\frac{1}{t^{(1)}_{k_a} - t^{(2)}_{m_a}} \Big(\prod_{j=1}^{n-1} \frac{t^{(1)}_{k_a} - y^{(j)}_3}{(t^{(1)}_{k_a} - y^{(j)}_1)(t^{(2)}_{m_a} - y^{(j)}_2)} \Big) \prod_{s=1}^{n} (t^{(2)}_{m_a} - z_s) \Big) . \tag{30}$$

6.6.2. Set $P_1(x) = \prod_{i=1}^{l_1} (t^{(1)}_i - x)$, $P_2(x) = \prod_{i=1}^{l_2} (t^{(2)}_i - x)$.

6.6.3. **Theorem.** *The canonical weight function and its degree decomposition are given by the following formula,*

$$\Psi(\boldsymbol{t}, \boldsymbol{z}, \boldsymbol{u}, \boldsymbol{y}) \ = \ u_1^{l_1} u_2^{l_1} \, \frac{\prod_{j=1}^{n-1} P_1(y^{(j)}_1) P_2(y^{(j)}_2)}{\prod_{s=1}^{n} P_1(z_s) P_2(z_s)} \ \times \ \times \Big[1 + \sum_{d=1}^{\min(l_1,l_2)} \frac{1}{d!} \big(\frac{u_3}{u_1 u_2} \big)^d \sum_{d-\mathrm{adm}\,(\boldsymbol{k},\boldsymbol{m})} \Xi_{(\boldsymbol{k},\boldsymbol{m})}(\boldsymbol{t}, \boldsymbol{z}, \boldsymbol{y}) \Big] . \tag{31}$$

The theorem is a direct corollary of the definition of the canonical weight function.

6.7. Comments on Theorem 6.6.3.

6.7.1. Here are the first terms of the degree decomposition

$$\Psi(\boldsymbol{t}, \boldsymbol{z}, \boldsymbol{u}, \boldsymbol{y}) = \Psi(\boldsymbol{t}, \boldsymbol{z}, \boldsymbol{u}, \boldsymbol{y})_0 + \Psi(\boldsymbol{t}, \boldsymbol{z}, \boldsymbol{u}, \boldsymbol{y})_1 + \Psi(\boldsymbol{t}, \boldsymbol{z}, \boldsymbol{u}, \boldsymbol{y})_2 + \dots \ .$$

We have

$$\Psi(\boldsymbol{t}, \boldsymbol{z}, \boldsymbol{u}, \boldsymbol{y})_0 \ = \ u_1^{l_1} u_2^{l_1} \, \frac{\prod_{j=1}^{n-1} P_1(y^{(j)}_1) P_2(y^{(j)}_2)}{\prod_{s=1}^{n} P_1(z_s) P_2(z_s)} \ ,$$

$$\Psi(\boldsymbol{t}, \boldsymbol{z}, \boldsymbol{u}, \boldsymbol{y})_1 = \Psi(\boldsymbol{t}, \boldsymbol{z}, \boldsymbol{u}, \boldsymbol{y})_0 \ \times \ \times \frac{u_3}{u_1 u_2} \sum_{k=1}^{l_1} \sum_{m=1}^{l_2} \frac{1}{t^{(1)}_k - t^{(2)}_m} \Big(\prod_{j=1}^{n-1} \frac{t^{(1)}_k - y^{(j)}_3}{(t^{(1)}_k - y^{(j)}_1)(t^{(2)}_m - y^{(j)}_2)} \Big) \prod_{s=1}^{n} (t^{(2)}_m - z_s) \ ,$$

and so on.

6.7.2. The degree zero term Ψ_0 of the canonical weight function is the product of functions of one variable. This is a manifestation of separation of variables.

6.7.3. The degree zero term Ψ_0 determines all other terms Ψ_d with $d > 0$.

Indeed, knowing Ψ_0 we know the roots $t^{(1)}_1, \dots, t^{(1)}_{l_1}$ of $P_1(x)$ and the roots $t^{(2)}_1, \dots, t^{(2)}_{l_2}$, of $P_2(x)$. Now for a d-admissible $(\boldsymbol{k}, \boldsymbol{m})$, the factor $\Psi_0\, \Xi_{(\boldsymbol{k},\boldsymbol{m})}$ has simple combinatorial meaning. Namely, we cross out d factors with indices in $(\boldsymbol{k}, \boldsymbol{m})$ from each P_1, P_2 entering Ψ_0 and then multiply the result by

$$\prod_{a=1}^{d} \frac{1}{t^{(1)}_{k_a} - t^{(2)}_{m_a}} \prod_{j=1}^{n-1} \frac{t^{(1)}_{k_a} - y^{(j)}_3}{\prod_{s=1}^{n} (t^{(1)}_{k_a} - z_s)} .$$

Notice that from $P_1(y^{(j)}_1)$ and $P_2(y^{(j)}_2)$ we cross out factors depending on $y^{(j)}_1, y^{(j)}_2$ and then multiply the result by terms depending on $y^{(j)}_3$. Hence the larger d, the more $y^{(j)}_3$-dependent factors participate in $\Psi_0\, \Xi_{(\boldsymbol{k},\boldsymbol{m})}$.

6.7.4. **Theorem.** *Assume that the numbers $\boldsymbol{z} = (z_1, \dots, z_n)$ are distinct and the numbers $\boldsymbol{t} = (t^{(1)}_1, \dots, t^{(1)}_{l_1},\ t^{(2)}_1, \dots, t^{(2)}_{l_2})$ are distinct. Assume that for any s, i, we have $z_s \notin \{t^{(i)}_1, \dots, t^{(i)}_{l_i}\}$. Assume that for such a $\boldsymbol{t}$, the canonical weight function $\Psi(\boldsymbol{t}, \boldsymbol{z}, \boldsymbol{u}, \boldsymbol{y})$, as a function of $\boldsymbol{u}, \boldsymbol{y}$, is an eigenvector of the Gaudin Hamiltonians. Then $\boldsymbol{t}$ is a solution to the Bethe ansatz equations (15).*

Proof. Let $H_i(\boldsymbol{z})\Psi(\boldsymbol{t}, \boldsymbol{z}, \boldsymbol{y}, u) = \mu_i \Psi(\boldsymbol{t}, \boldsymbol{z}, \boldsymbol{y}, u)$ for $i = 1, \dots, n$.

Since $\Psi(\boldsymbol{t}, \boldsymbol{z}, \boldsymbol{y}, u)$ is an eigenfunction of the Gaudin Hamiltonians, it is annihilated by the corresponding operators $K_j(\boldsymbol{z})$, $j = 1, \dots, n-1$. Hence the degree zero term $(K_j(\boldsymbol{z})\Psi(\boldsymbol{t}, \boldsymbol{z}, \boldsymbol{y}, u))_0$ is zero. Hence $(K_j(\boldsymbol{z})_0 \Psi(\boldsymbol{t}, \boldsymbol{z}, \boldsymbol{y}, u))_0|_{\mathfrak{D}}$ is zero by Lemma 6.5.3.

Using $\boldsymbol{t}$, define $P_1(x), P_2(x)$ as in Section 6.6.2. Then the function $f = P_1(y^{(j)}_1)P_2(y^{(j)}_1)$ satisfies equation (29) by Theorem 6.5.4. Then $\boldsymbol{t}$ is a solution to the Bethe ansatz equations (15) by Theorem 4.2.3. □

6.7.5. Theorem 3.4.1 says that if $\boldsymbol{t}$ is a solution to the Bethe ansatz equations, then the value at $\boldsymbol{t}$ of the canonical weight function is an eigenvector of the Gaudin Hamiltonians and is a singular vector. Theorem 6.7.4 gives a converse statement: if the value of the canonical weight function at some point $\boldsymbol{t}$ is an eigenvector of the Gaudin Hamiltonians, then $\boldsymbol{t}$ is a solution to the Bethe ansatz equations and that value is a singular vector.

6.8. It would be good to have for $\mathfrak{g} = \mathfrak{sl}_3$ an analog of Lemma 5.5.

7. Appendix. New form of the Bethe ansatz equations for general Kac-Moody algebras

7.1. **Kac-Moody algebras.** Let $A = (a_{ij})^r_{i,j=1}$ be a generalized Cartan matrix, $a_{ii} = 2$, $a_{ij} = 0$ if and only $a_{ji} = 0$, $a_{ij} \in \mathbb{Z}_{\le 0}$ if $i \ne j$. We assume that A is symmetrizable, i.e. there is a diagonal matrix $D = \operatorname{diag}(d_1, \dots, d_r)$ with positive integers d_i such that $B = DA$ is symmetric.

Let $\mathfrak{g} = \mathfrak{g}(A)$ be the corresponding complex Kac-Moody Lie algebra (see [K], §1.2), $\mathfrak{h} \subset \mathfrak{g}$ the Cartan subalgebra. The associated scalar product is non-degenerate on $\mathfrak{h}^*$ and $\dim \mathfrak{h} = r + 2d$, where d is the dimension of the kernel of the Cartan matrix A.

Let $\alpha_i \in \mathfrak{h}^*$, $\alpha_i^\vee \in \mathfrak{h}$, $i = 1, \dots, r$, be the sets of simple roots, coroots, respectively. We have

$$(\alpha_i, \alpha_j) \;=\; d_i\, a_{ij}\ , \qquad \langle \lambda, \alpha_i^\vee \rangle \;=\; 2(\lambda, \alpha_i)/(\alpha_i, \alpha_i) \;\text{ for }\; \lambda \in \mathfrak{h}^*\ .$$

7.2. **Bethe ansatz equations for the Gaudin model.** Let $\boldsymbol{z} = (z_1, \dots, z_n)$ be a collection of distinct complex numbers. Let $\boldsymbol{\Lambda} = (\Lambda_1, \dots, \Lambda_n)$, $\Lambda_s \in \mathfrak{h}^*$, be a collection of $\mathfrak{g}$-weights and $\boldsymbol{l} = (l_1, \dots, l_r)$ a collection of nonnegative integers. Set $l = l_1 + \dots + l_r$ and

$$\boldsymbol{t} = (t_1^{(1)}, \dots, t_{l_1}^{(1)}, \dots, t_1^{(r)}, \dots, t_{l_r}^{(r)})\ .$$

The Bethe ansatz equations for the Gaudin model associated with this data is the following system of algebraic equations with respect to $\boldsymbol{t}$:

$$-\sum_{s=1}^{n} \frac{(\Lambda_s, \alpha_i)}{t_j^{(i)} - z_s} \;+\; \sum_{s,\ s\neq i} \sum_{k=1}^{l_s} \frac{(\alpha_s, \alpha_i)}{t_j^{(i)} - t_k^{(s)}} \;+\; \sum_{s,\ s\neq j} \frac{(\alpha_i, \alpha_i)}{t_j^{(i)} - t_s^{(i)}} \;=\; 0\ , \tag{32}$$

where $i = 1, \dots, r$, $j = 1, \dots, l_i$.

7.3. **The second form of the Bethe ansatz equations.** For a function of x we write $f' = df/dx$ and $\ln'(f) = f'/f$.

For given $\boldsymbol{z}, \boldsymbol{\Lambda}, \boldsymbol{t}$ and $i = 1, \dots, r$, we introduce polynomials

$$P_i(x) \;=\; \prod_{j=1}^{l_i} (t_j^{(i)} - x)\ , \qquad T_i(x) \;=\; \prod_{s=1}^{n} (x - z_s)^{\langle \Lambda_s, \alpha_i^\vee \rangle}\ ,$$

$$F_i(x) \;=\; \Big(\prod_{s=1}^{n} (x - z_s)\Big) \prod_{j,\ a_{ij}<0} P_j(x)\ , \qquad G_i(x) \;=\; F_i(x)\, \ln' \Big(T_i(x) \prod_{j,\ j\neq i} P_j(x)^{-\langle \alpha_j, \alpha_i^\vee \rangle} \Big)\ .$$

7.3.1. **Lemma ([MV1]).** *Assume that the roots $\boldsymbol{t}$ of polynomials $P_1, \dots, P_r$ are all simple, distinct and different from $z_1, \dots, z_n$.*

Then $\boldsymbol{t}$ is a solution of the Bethe ansatz equations (32) if and only if for every $i = 1, \dots, r$, the polynomial $F_i P_i'' - G_i P'$ is divisible by the polynomial P_i.

In other words, the roots of $P_1, \dots, P_r$ form a solution of the Bethe ansatz equations (32) if and only if for every $i = 1, \dots, r$, there exists a polynomial H_i of degree not greater than deg $F_i - 2$ *such that P_i is a solution to the differential equation*

$$F_i\, P_i'' \;-\; G_i\, P_i' \;+\; H_i\, P_i \;=\; 0\ . \tag{33}$$

7.4. **The new form of the Bethe ansatz equations.**

7.4.1. **Theorem.** *Assume that the roots* $\boldsymbol{t}$ *of polynomials* $P_1, \dots, P_r$ *are all simple, distinct and different from* $z_1, \dots, z_n$*. Then* $\boldsymbol{t}$ *is a solution of the Bethe ansatz equations (32) if and only if there exist numbers* $\mu_1, \dots, \mu_n$*,* $\mu_1 + \dots + \mu_n = 0$*, such that*

$$\sum_{i=1}^{r}(\alpha_i,\alpha_i)\,\frac{P_i''}{P_i} \;+\; \sum_{i\neq j}(\alpha_i,\alpha_j)\,\frac{P_i'P_j'}{P_iP_j} - \sum_{i=1}^{r}(\alpha_i,\alpha_i)\,\frac{T_i'P_i'}{T_iP_i} \;+ \\ +\;\sum_{s=1}^{n}\frac{1}{x-z_s}\big(\,\mu_j - \sum_{k\neq j}\frac{(\Lambda_j,\Lambda_k)}{z_j-z_k}\,\big) \;=\; 0\;. \tag{34}$$

This fact may be considered as a generalization of Stieltjes' Lemma 4.1.2 (see also [Sti], and Sec. 6.8 in [Sz]) to an arbitrary Kac-Moody algebra.

Proof. Let

$$F(x) \;=\; \prod_{s=1}^{n}(x-z_s)\;, \qquad P(x) \;=\; \prod_{i=1}^{r}P_i(x)\;.$$

Let us show that (34) implies (32). Multiply (34) by P and substitute for x a root of $P_i,\ x = t_s^{(i)}$. Then

$$\frac{(\alpha_i,\alpha_i)}{2}\frac{P_i''(t_s^{(i)})}{P_i'(t_s^{(i)})} \;+\; \sum_{j,j\neq i}(\alpha_i,\alpha_j)\frac{P_j'(t_s^{(i)})}{P_j(t_s^{(i)})} \;-\; \sum_{a=1}^{n}\frac{(\Lambda_a,\alpha_i)}{t_s^{(i)}-z_a} \;=\; 0\;.$$

This is exactly the $t_s^{(i)}$-th equation of (32).

Let us show the converse. If the zeros of $P_1, \dots, P_r$ form a solution to (32), then for $i = 1, \dots, r$ by Lemma 7.3.1, we have

$$\Big(\frac{(\alpha_i,\alpha_i)}{2}\frac{P_i''}{P_i} + \sum_{a,a\neq i}(\alpha_i,\alpha_a)\frac{P_i'P_a'}{P_iP_a} - \frac{(\alpha_i,\alpha_i)}{2}\frac{T_i'P_i'}{T_iP_i}\Big)FP \;+\; \tilde{H}_iP_i \;=\; 0\;, \tag{35}$$

where $\tilde{H}_i$ is some polynomial.

Add all these equations. Then (34) is proved if we show that

$$\sum_{i=1}^{r}\tilde{H}_iP_i \;+\; \sum_{i<j}(\alpha_i,\alpha_j)\frac{P_i'P_j'}{P_iP_j}PF \tag{36}$$

is divisible by P. To show that it suffices to show that the expression in (36) is divisible by $P_1, \dots, P_r$. Let us show that it is divisible by P_1.

Indeed the sum

$$\tilde{H}_1P_1 \;+\; \sum_{1<i<j}(\alpha_i,\alpha_j)\frac{P_i'P_j'}{P_iP_j}PF$$

is divisible by P_1. Now for $i \neq 1$, from the i-th equation in (35) we obtain that

$$\tilde{H}_iP_i \;+\; (\alpha_i,\alpha_1)\frac{P_i'P_1'}{P_iP_1}PF$$

is divisible by P_1. Hence the expression in (36) is divisible by P_1.

Note that the quotient of (36) by P is a polynomial of degree at most $n-2$ and therefore the quotient of (36) by PF has the form

$$\sum_{s=1}^{n} \frac{1}{x-z_s}\Big(\mu_j - \sum_{k\neq j} \frac{(\Lambda_j, \Lambda_k)}{z_j - z_k} \Big)$$

for some numbers $\mu_1, \dots, \mu_n$ with $\mu_1 + \dots + \mu_n = 0$. $\square$

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